

HIGHER COHOMOLOGIES OF COMMUTATIVE MONOIDS

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ABSTRACT. Extending Eilenberg-Mac Lane's methods, higher level cohomologies for commutative monoids are introduced and studied. Relationships with pre-existing theories (Leech, Grillet, etc.) are stated. The paper includes a cohomological classification for symmetric monoidal groupoids and explicit computations for cyclic monoids.

1. INTRODUCTION AND SUMMARY

In [19, Chapter X, §12], Mac Lane explains how to define, for each integer $r \geq 0$, the r th level cohomology groups of a (skew) commutative DGA-algebra (differential graded augmented algebra) over a commutative ring K , say D : Take the commutative DGA-algebra $\mathbf{B}^r(D)$, obtained by iterating r times the reduced bar construction on D , and then, for any K -module A , define

$$H^n(D, r; A) = H^n(\mathrm{Hom}_K(\mathbf{B}^r(D), A), \quad n = 0, 1, \dots,$$

where $\mathrm{Hom}_K(\mathbf{B}^r(D), A)$ is the cochain complex obtained by applying the functor $\mathrm{Hom}_K(-, A)$ to the underlying chain complex of K -modules $\mathbf{B}^r(D)$.

This process may be applied, for example, when $D = \mathbb{Z}G$ is the group ring of an abelian group G , regarded as a trivially graded DGA-ring, augmented by $\alpha : \mathbb{Z}G \rightarrow \mathbb{Z}$ with $\alpha(x) = 1$ for all $x \in G$. Thus, the Eilenberg-Mac Lane r th level cohomology groups of the abelian group G with coefficients in an abelian group A are defined by

$$(1) \quad H^n(G, r; A) = H^n(\mathbb{Z}G, r; A).$$

In particular, the first level cohomology groups $H^n(G, 1; A) = H^n(G, A)$ are the ordinary cohomology groups of G with coefficients in the trivial G -module A [19, Chapter IV, Corollary 5.2]. These r th level cohomology groups of abelian groups were studied primarily with interest in Algebraic Topology. For instance, they have a topological interpretation in terms of the Eilenberg-Mac Lane spaces $K(G, r)$, owing to the isomorphisms $H^n(G, r; A) \cong H^n(K(G, r), A)$ [7, Theorem 20.3]. However, they early found application in solving purely algebraic problems. For example, we could recall that central group extensions of G by A are classified by cohomology classes in $H^2(G, 1; A)$, while abelian group extensions of G by A are classified by cohomology classes in $H^3(G, 2; A)$ [8, §26, (26.2), (26.3)]; or that second level cohomology classes in $H^4(G, 2; A)$ classify braided monoidal categorical groups [14, Theorem 3.3], while third level cohomology classes in $H^5(G, 3; A)$ classify Picard categories [21, II, Proposition 5].

Here, we introduce a generalization of Eilenberg-Mac Lane's theory for abelian groups to commutative monoids. The obtained r th level cohomology groups of a commutative monoid M , denoted by

$$H^n(M, r; \mathcal{A}),$$

enjoy many desirable properties, whose study this work and its companion paper [5] are mainly dedicated to. In our development, the role of coefficients is now played by abelian group objects \mathcal{A} in the comma category of commutative monoids over M . We call them $\mathbb{H}M$ -modules since,

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as a result by Grillet [12, Chapter XII, §2], they are the same as abelian group valued functors $\mathcal{A} : \mathbb{H}M \rightarrow \mathbf{Ab}$ on the small category $\mathbb{H}M$, whose set of objects is M and set of arrows $M \times M$, with $(x, y) : x \rightarrow xy$.

For any given commutative monoid M , the category of chain complexes of $\mathbb{H}M$ -modules is an abelian category. In Section 2, we show that it is also a symmetric monoidal category, with a distributive tensor product $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$, and whose unit object is \mathbb{Z} , the concentrated in degree zero complex defined by the constant $\mathbb{H}M$ -module given by the abelian group \mathbb{Z} of integers. Hence, commutative *DGA-algebras over $\mathbb{H}M$* arise as internal commutative monoids \mathcal{A} in the symmetric monoidal category of complexes of $\mathbb{H}M$ -modules, endowed with a morphism of internal monoids $\mathcal{A} \rightarrow \mathbb{Z}$.

Quite similarly as for ordinary commutative DGA-algebras over a commutative ring, a reduced bar construction $\mathcal{A} \mapsto \mathbf{B}(\mathcal{A})$ works on these DGA-algebras over $\mathbb{H}M$. Thus, $\mathbf{B}(\mathcal{A})$ is obtained from \mathcal{A} by first totalizing the double complex of $\mathbb{H}M$ -modules

$$\bigoplus_{p \geq 0} \mathcal{A} / \mathbb{Z} \otimes_{\mathbb{H}M} \overset{(p \text{ factors})}{\cdots} \otimes_{\mathbb{H}M} \mathcal{A} / \mathbb{Z},$$

and then enriching the (suitably graded) totalized complex of $\mathbb{H}M$ -modules with a multiplicative structure by a shuffle product. We do this in Section 3, where we also define, for any $\mathbb{H}M$ -module \mathcal{B} , the r th level cohomology groups of \mathcal{A} with coefficients in \mathcal{B} by

$$H^n(\mathcal{A}, r; \mathcal{B}) = H^n(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}), \mathcal{B})), \quad n = 0, 1, \dots$$

Next, in Section 4 we briefly study free $\mathbb{H}M$ -modules. These arise as a left adjoint construction to a forgetful functor from the category of $\mathbb{H}M$ -modules to the comma category of sets over the underlying set of M . In particular, in Section 5 we introduce the free $\mathbb{H}M$ -module on the identity map $id_M : M \rightarrow M$, denoted by $\mathcal{Z}M$. This becomes a (trivially graded) commutative DGA-algebra over $\mathbb{H}M$ and then, for each integer positive r , we define the r th level cohomology groups of a commutative monoid M with coefficients in a $\mathbb{H}M$ -module \mathcal{A} by

$$(2) \quad H^n(M, r; \mathcal{A}) = H^n(\mathcal{Z}M, r; \mathcal{A}).$$

When $M = G$ is an abelian group, for any integer $r \geq 0$, $\mathbf{B}^r(\mathcal{Z}G)$ is isomorphic to the constant DGA-algebra over $\mathbb{H}G$ defined by the Eilenberg-Mac Lane DGA-ring $\mathbf{B}^r(\mathbb{Z}G) (= A_N(G, r)$ in [7, §14]). Hence, for any abelian group A , viewed as a constant $\mathbb{H}G$ -module, the cohomology groups $H^n(G, r; A)$ defined as in (2) are naturally isomorphic to those by Eilenberg and Mac Lane in (1), which, recall, compute the cohomology groups of the spaces $K(G, r)$ as $H^n(G, r; A) \cong H^n(K(G, r), A)$. In the companion paper [5] we show that, for any commutative monoid M , there are isomorphisms

$$H^n(M, r; \mathcal{A}) \cong H^n(\overline{W}^r M, \mathcal{A}),$$

where $H^n(\overline{W}^r M, \mathcal{A})$, $n \geq 0$, are Gabriel-Zisman cohomology groups [10, Appendix II] of the underlying simplicial set of the simplicial monoid $\overline{W}^r M$, obtained by iterating the \overline{W} construction on the constant simplicial monoid defined by M .

An analysis of the complex $\mathbf{B}(\mathcal{Z}M)$, for M any commutative monoid, leads us in Proposition 5.3 to identify the cohomology groups $H^n(M, 1; \mathcal{A})$ with the standard cohomology groups $H_L^n(M, \mathcal{A})$ by Leech [16]. Recall that Leech cohomology groups of a (not necessarily commutative) monoid M take coefficients in abelian group valued functors on the category $\mathbb{D}M$, whose objects are the elements of M and arrows triples $(x, y, z) : y \rightarrow xyz$. When the monoid M is commutative, there is a natural functor $\mathbb{D}M \rightarrow \mathbb{H}M$ which is the identity on objects and carries a morphism (x, y, z) of $\mathbb{D}M$ to the morphism (y, xz) of $\mathbb{H}M$. Via this functor, every $\mathbb{H}M$ -module \mathcal{A} is regarded as a $\mathbb{D}M$ -module and we prove that, for any commutative monoid M and $\mathbb{H}M$ -module \mathcal{A} , there are natural isomorphisms

$$H^n(M, 1; \mathcal{A}) \cong H_L^n(M, \mathcal{A}), \quad n = 0, 1, \dots$$

For any $r \geq 2$, we show explicit descriptions of the complexes $\mathbf{B}^r(\mathcal{Z}M)$ truncated at dimensions $\leq r + 3$, which are useful both for theoretical and computational interests concerning the cohomology groups $H^n(M, r; \mathcal{A})$ for $n \leq r + 2$. Some conclusions here summarize as follows:

- $H^0(M, r; \mathcal{A}) \cong H^0(M, 1; \mathcal{A}) \cong H_L^0(M, \mathcal{A}) \cong \mathcal{A}(e)$,

where $\mathcal{A}(e)$ is the abelian group attached by \mathcal{A} at the identity e of the monoid.

- $H^n(M, r; \mathcal{A}) = 0$, for $0 < n < r$,
- $H^r(M, r; \mathcal{A}) \cong H^1(M, 1; \mathcal{A}) \cong H_L^1(M, \mathcal{A}) \cong H_G^1(M, \mathcal{A})$,
- $H^{r+1}(M, r; \mathcal{A}) \cong H^3(M, 2; \mathcal{A}) \cong H_G^2(M, \mathcal{A})$.

where $H_G^n(M, \mathcal{A})$ denotes the n -th cohomology group by Grillet [11, 12].

- $H^4(M, 2; \mathcal{A}) \cong H_C^3(M, \mathcal{A})$,

where $H_C^3(M, \mathcal{A})$ is the third commutative cohomology group by the authors in [2].

- $H^{r+2}(M, r; \mathcal{A}) \cong H^5(M, 3; \mathcal{A})$, for $r \geq 3$.
- There are natural inclusions $H_C^3(M, \mathcal{A}) \subseteq H^5(M, 3; \mathcal{A}) \subseteq H_C^3(M, \mathcal{A})$.

Most of these cohomology groups above have known algebraic interpretations. For example, elements of $H^1(M, 1; \mathcal{A}) = H_L^1(M, \mathcal{A})$ are *derivations* [16, Chapter II, 2.7]. Cohomology classes in $H^2(M, 1; \mathcal{A}) = H_L^2(M, \mathcal{A})$ are isomorphism classes of *group coextensions* [16, Chapter V, §2] (or [22, Theorem 2]), while elements of $H^3(M, 2; \mathcal{A}) = H_G^2(M, \mathcal{A})$ classify *abelian group coextensions* [12, Chapter V, §4]. Cohomology classes in $H^3(M, 1; \mathcal{A}) = H_L^3(M, \mathcal{A})$ are equivalence classes of *monoidal abelian groupoids* [1, Theorem 4.3], elements of $H^4(M, 2; \mathcal{A}) = H_C^3(M, \mathcal{A})$ are equivalence classes of *braided monoidal abelian groupoids* [2, Theorem 4.5], and elements of $H_C^3(M, \mathcal{A})$ are equivalence classes of *strictly commutative monoidal abelian groupoids* [3, Theorem 3.1]. Thus, among them, only the cohomology groups $H^5(M, 3; \mathcal{A})$ are pending of interpretation, and we solve this in Section 6. Here we give a natural interpretation of the cohomology classes in $H^5(M, 3; \mathcal{A})$ in terms of equivalence classes of *symmetric monoidal abelian groupoids*, that is, groupoids \mathcal{M} , whose isotropy groups $\text{Aut}_{\mathcal{M}}(x)$ are all abelian, endowed with a monoidal structure by a tensor functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, a unit object I , and coherent associativity, unit and commutativity constraints $\alpha : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$, $\iota : I \otimes x \cong x$, and $c : x \otimes y \cong y \otimes x$ which satisfy the symmetry condition $c^2 = id$. The classification of symmetric monoidal abelian groupoids we give extends that, above refereed, by Sinh in [21, II, Proposition 5] for Picard categories.

In last Section 7, we compute the cohomology groups $H^n(M, r; \mathcal{A})$, for $n \leq r + 2$, when M is any cyclic monoid.

2. COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS OVER $\mathbb{H}M$

Throughout this paper M denotes a *commutative multiplicative monoid*, whose unit is e .

As noted in the introduction, in [12, Chapter XII, §2] Grillet observes that the category of abelian group objects in the slice category of commutative monoids over M , $\mathbf{CMon} \downarrow_M$, is equivalent to the category of abelian group valued functors on the small category $\mathbb{H}M$, whose object set is M and arrow set $M \times M$, where $(x, y) : x \rightarrow xy$. Composition is given by $(xy, z)(x, y) = (x, yz)$, and the identity of an object x is (x, e) . The category of functors from $\mathbb{H}M$ into the category of abelian groups will be denoted by

$$\mathbb{H}M\text{-Mod}$$

and called the category of *$\mathbb{H}M$ -modules*. An $\mathbb{H}M$ -module, say $\mathcal{A} : \mathbb{H}M \rightarrow \mathbf{Ab}$, then consists of abelian groups $\mathcal{A}(x)$, one for each $x \in M$, and homomorphisms $y_* : \mathcal{A}(x) \rightarrow \mathcal{A}(xy)$, one for each $x, y \in M$, such that, for any $x, y, z \in M$, $y_* z_* = (yz)_* : \mathcal{A}(x) \rightarrow \mathcal{A}(xyz)$ and, for any $x \in M$, $e_* = id_{\mathcal{A}(x)} : \mathcal{A}(x) \rightarrow \mathcal{A}(xe) = \mathcal{A}(x)$.

For instance, let

$$(3) \quad \mathbb{Z} : \mathbb{H}M \rightarrow \mathbf{Ab}, \quad x \mapsto \mathbb{Z}(x) = \mathbb{Z}\{x\},$$

be the $\mathbb{H}M$ -module which associates to each element $x \in M$ the free abelian group on the generator x , and to each pair (x, y) the isomorphism of abelian groups $y_* : \mathbb{Z}(x) \rightarrow \mathbb{Z}(xy)$ given on the generator by $y_*x = xy$. This is isomorphic to the $\mathbb{H}M$ -module defined by the constant functor on $\mathbb{H}M$ which associates the abelian group of integers \mathbb{Z} to any $x \in M$.

For two $\mathbb{H}M$ -modules \mathcal{A} and \mathcal{B} , a morphism between them (i.e., a natural transformation) $f : \mathcal{A} \rightarrow \mathcal{B}$ consists of homomorphisms $f_x : \mathcal{A}(x) \rightarrow \mathcal{B}(x)$, such that, for any $x, y \in M$, the square below commutes.

$$\begin{array}{ccc} \mathcal{A}(x) & \xrightarrow{f_x} & \mathcal{B}(x) \\ y_* \downarrow & & \downarrow y_* \\ \mathcal{A}(xy) & \xrightarrow{f_{xy}} & \mathcal{B}(xy) \end{array}$$

The category of $\mathbb{H}M$ -modules is abelian and we refer to [19, Chapter IX, §3] for details. Recall that the set of morphisms between two $\mathbb{H}M$ -modules \mathcal{A} and \mathcal{B} , denoted by $\text{Hom}_{\mathbb{H}M}(\mathcal{A}, \mathcal{B})$, is an abelian group by objectwise addition, that is, if $f, g : \mathcal{A} \rightarrow \mathcal{B}$ are morphisms, then $f + g : \mathcal{A} \rightarrow \mathcal{B}$ is defined by setting $(f + g)_x = f_x + g_x$, for each $x \in M$. The zero $\mathbb{H}M$ -module is the constant functor $0 : \mathbb{H}M \rightarrow \mathbf{Ab}$ defined by the trivial abelian group 0 , and the direct sum of two $\mathbb{H}M$ -modules \mathcal{A} and \mathcal{B} is given by taking direct sum at each object, that is, $(\mathcal{A} \oplus \mathcal{B})(x) = \mathcal{A}(x) \oplus \mathcal{B}(x)$. Similarly, all limits and colimits (in particular, kernels, images, cokernels, etc.) in the category $\mathbb{H}M\text{-Mod}$ are pointwise constructed.

Remark 2.1. Every abelian group A defines a *constant* $\mathbb{H}M$ -module, equally denoted by A , such that $A(x) = A$ and $y_* = \text{id}_A : A(x) \rightarrow A(xy)$, for any $x, y \in M$. In this way, the category of abelian groups becomes a full subcategory of the category of $\mathbb{H}M$ -modules.

When $M = G$ is an abelian group, then this inclusion $\mathbf{Ab} \hookrightarrow \mathbb{H}G\text{-Mod}$ is actually an equivalence of categories. In the other direction, we have the functor associating to each $\mathbb{H}G$ -module \mathcal{A} the abelian group $\mathcal{A}(e)$, and there is natural isomorphism of $\mathbb{H}G$ -modules $\mathcal{A} \cong \mathcal{A}(e)$ whose component at each $x \in G$ is the isomorphism of abelian groups $x_*^{-1} : \mathcal{A}(x) \rightarrow \mathcal{A}(e)$.

2.1. Tensor product of $\mathbb{H}M$ -modules. For any two $\mathbb{H}M$ -modules \mathcal{A}, \mathcal{B} , their *tensor product*, denoted by $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$, is the $\mathbb{H}M$ -module defined as follows: It attaches to any $x \in M$ the abelian group defined by the coequalizer sequence of homomorphisms

$$\bigoplus_{uvw=x} \mathbb{Z}(u) \otimes \mathcal{A}(v) \otimes \mathcal{B}(w) \xrightleftharpoons[\psi]{\phi} \bigoplus_{zt=x} \mathcal{A}(z) \otimes \mathcal{B}(t) \longrightarrow (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x),$$

where, for any two abelian groups A and B , $A \otimes B$ denotes their tensor product as \mathbb{Z} -modules, the direct sum on the left is taken over all triples $(u, v, w) \in M^3$ such that $uvw = x$, the direct sum on the middle is over all pairs $(z, t) \in M^2$ with $zt = x$, and the homomorphisms ϕ and ψ are defined by

$$\begin{aligned} \phi(u \otimes a_v \otimes b_w) &= u_* a_v \otimes b_w \in \mathcal{A}(uv) \otimes \mathcal{B}(w), \\ \psi(u \otimes a_v \otimes b_w) &= a_v \otimes u_* b_w \in \mathcal{A}(v) \otimes \mathcal{B}(uw), \end{aligned}$$

for all $u, v, w \in M$ with $uvw = x$, $a_v \in \mathcal{A}(v)$, and $b_w \in \mathcal{B}(w)$. For any pair $(x, y) \in M^2$, the homomorphism

$$y_* : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x) \rightarrow (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(xy)$$

is given on generators by

$$y_*(a_z \otimes b_t) = y_* a_z \otimes b_t = a_z \otimes y_* b_t, \quad (a_z \in \mathcal{A}(z), b_t \in \mathcal{B}(t), zt = x).$$

If $f : \mathcal{A} \rightarrow \mathcal{A}'$ and $g : \mathcal{B} \rightarrow \mathcal{B}'$ are morphisms of $\mathbb{H}M$ -modules, then there is an induced one $f \otimes g : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} \rightarrow \mathcal{A}' \otimes_{\mathbb{H}M} \mathcal{B}'$ such that, for each $x \in M$, the homomorphism

$$(f \otimes g)_x : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x) \rightarrow (\mathcal{A}' \otimes_{\mathbb{H}M} \mathcal{B}')(x)$$

is given on generators by

$$(f \otimes g)_x(a_z \otimes b_t) = f_z a_z \otimes g_t b_t, \quad (a_z \in \mathcal{A}(z), b_t \in \mathcal{B}(t), zt = x).$$

Thus, we have a distributive tensor functor

$$- \otimes_{\mathbb{H}M} - : \mathbb{H}M\text{-Mod} \times \mathbb{H}M\text{-Mod} \rightarrow \mathbb{H}M\text{-Mod}.$$

Further, there are canonical isomorphisms of $\mathbb{H}M$ -modules

$$\begin{aligned} l_{\mathcal{A}} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathcal{A} &\cong \mathcal{A}, & c_{\mathcal{A}, \mathcal{B}} : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} &\cong \mathcal{B} \otimes_{\mathbb{H}M} \mathcal{A}, \\ a_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : \mathcal{A} \otimes_{\mathbb{H}M} (\mathcal{B} \otimes_{\mathbb{H}M} \mathcal{C}) &\cong (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) \otimes_{\mathbb{H}M} \mathcal{C}, \end{aligned}$$

respectively defined by the formulas

$$\begin{aligned} l_{zt}(z \otimes a_t) &= z_* a_t, & c_{zt}(a_z \otimes b_t) &= b_t \otimes a_z, \\ a_{yzt}(a_y \otimes (b_z \otimes c_t)) &= (a_y \otimes b_z) \otimes c_t, \end{aligned}$$

which are easily proven to be natural and coherent in the sense of [17, Theorem 5.1]. Therefore, $\mathbb{H}M\text{-Mod}$ is a symmetric monoidal category. We will usually treat the constraints above as identities, so we think of $\mathbb{H}M\text{-Mod}$ as a symmetric strict monoidal category.

2.2. Tensor product of complexes of $\mathbb{H}M$ -modules. The (positive) complexes of $\mathbb{H}M$ -modules

$$\mathcal{A} = \cdots \rightarrow \mathcal{A}_2 \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0$$

and the morphisms between them also form an abelian symmetric monoidal category, where the tensor product $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$ of two complexes of $\mathbb{H}M$ -modules \mathcal{A} and \mathcal{B} is the graded $\mathbb{H}M$ -module whose component of degree n is

$$(\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_n = \bigoplus_{p+q=n} \mathcal{A}_p \otimes_{\mathbb{H}M} \mathcal{B}_q,$$

and whose differential ∂^\otimes , at any $x \in M$,

$$\partial_x^\otimes : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_n(x) \rightarrow (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_{n-1}(x),$$

is defined on generators by the Leibniz formula

$$\partial_x^\otimes(a_z \otimes b_t) = \partial_z a_z \otimes b_t + (-1)^p a_z \otimes \partial_t b_t.$$

for all $z, t \in M$ such that $zt = x$, $a_z \in \mathcal{A}_p(z)$, $b_t \in \mathcal{B}_q(t)$, and $p, q \geq 0$ such that $p + q = n$.

In this monoidal category, the unit object is \mathbb{Z} , defined in (3), regarded as a complex concentrated in degree zero. The structure constraints

$$(4) \quad \begin{aligned} l_{\mathcal{A}} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathcal{A} &\cong \mathcal{A}, & c_{\mathcal{A}, \mathcal{B}} : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} &\cong \mathcal{B} \otimes_{\mathbb{H}M} \mathcal{A}, \\ a_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : \mathcal{A} \otimes_{\mathbb{H}M} (\mathcal{B} \otimes_{\mathbb{H}M} \mathcal{C}) &\cong (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) \otimes_{\mathbb{H}M} \mathcal{C}, \end{aligned}$$

are respectively defined by the formulas

$$\begin{aligned} l_{xy}(x \otimes a_y) &= x_* a_y, \\ c_{xy}(a_x \otimes b_y) &= (-1)^{pq} b_y \otimes a_x, \\ a_{xyz}(a_x \otimes (b_y \otimes c_z)) &= (a_x \otimes b_y) \otimes c_z, \end{aligned}$$

for any $x, y, z \in M$, $a_x \in \mathcal{A}_p(x)$, $b_y \in \mathcal{B}_q(y)$, and $c_z \in \mathcal{C}_r(z)$. As for $\mathbb{H}M$ -modules, we will treat these constraints as identities.

2.3. Commutative differential graded algebras over $\mathbb{H}M$. A commutative *differential graded algebra* (DG-algebra) \mathcal{A} over $\mathbb{H}M$ is defined to be a commutative monoid in the symmetric monoidal category of complexes of $\mathbb{H}M$ -modules, see [18, Chapter VII, §3]. Hence, \mathcal{A} is a complex of $\mathbb{H}M$ -modules equipped with a *multiplication morphism* of complexes $\circ : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{A} \rightarrow \mathcal{A}$ satisfying the associativity $\circ(\circ \otimes id) = \circ(id \otimes \circ)$ and the commutativity $\circ \mathbf{c} = \circ$, and a *unit morphism* of complexes $\iota : \mathbb{Z} \rightarrow \mathcal{A}$ satisfying $\circ(\iota \otimes id_{\mathcal{A}}) = \mathbf{l}_{\mathcal{A}}$. We write

$$1 = \iota_e(e) \in \mathcal{A}_0(e)$$

and, for any $x, y \in M$, $a_x \in \mathcal{A}_p(x)$, and $a_y \in \mathcal{A}_q(y)$,

$$a_x \circ a_y = \circ_{xy}(a_x \otimes a_y) \in \mathcal{A}_{p+q}(xy),$$

so that the algebra structure on the complex \mathcal{A} gives us multiplication homomorphisms of abelian groups

$$\mathcal{A}_p(x) \otimes \mathcal{A}_q(y) \rightarrow \mathcal{A}_{p+q}(xy), \quad a_x \otimes a_y \mapsto a_x \circ a_y,$$

and a *unit* $1 \in \mathcal{A}_0(e)$, satisfying

$$(5) \quad x_* a_y \circ a_z = x_*(a_y \circ a_z) = a_y \circ x_* a_z,$$

$$(6) \quad a_x \circ a_y = (-1)^{pq} a_y \circ a_x,$$

$$(7) \quad 1 \circ a_x = a_x = a_x \circ 1,$$

$$(8) \quad a_x \circ (a_y \circ a_z) = (a_x \circ a_y) \circ a_z,$$

$$(9) \quad \partial_{xy}(a_x \circ a_y) = \partial_x a_x \circ a_y + (-1)^p a_x \circ \partial_y a_y,$$

for all $x, y, z \in M$, $a_x \in \mathcal{A}_p(x)$, $a_y \in \mathcal{A}_q(y)$, and $a_z \in \mathcal{A}_r(z)$.

In these terms, a morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ of commutative DG-algebras over $\mathbb{H}M$ is a morphism of complexes of $\mathbb{H}M$ -modules such that $f_{xy}(a_x \circ a_y) = f_x a_x \circ f_y a_y$, and $f_e(1) = 1$.

The category of commutative DG-algebras over $\mathbb{H}M$ is symmetric monoidal. The tensor product of two of them $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$ is given by their tensor product as complexes of $\mathbb{H}M$ -modules endowed with multiplication such that, for $u, v, x, y \in M$, $a_u \in \mathcal{A}_p(u)$, etc.,

$$(a_u \otimes b_x) \circ (a_y \otimes b_z) = (a_u \circ a_y) \otimes (b_x \circ b_z)$$

and with unit $1 \otimes 1 \in (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_0(e)$. Observe that the canonical isomorphisms in (4) are actually of DG-algebras whenever the data \mathcal{A} , \mathcal{B} and \mathcal{C} therein are DG-algebras over $\mathbb{H}M$.

Commutative DG-algebras over $\mathbb{H}M$ which are concentrated in degree zero are the same as commutative monoids in the symmetric monoidal category of $\mathbb{H}M$ -modules, and they are simply called *algebras over $\mathbb{H}M$* or *$\mathbb{H}M$ -algebras*. For example, \mathbb{Z} is an $\mathbb{H}M$ -algebra with multiplication the unit constraint $\mathbf{l} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathbb{Z} \cong \mathbb{Z}$ and unit the identity $id : \mathbb{Z} \rightarrow \mathbb{Z}$. In other words, \mathbb{Z} is an $\mathbb{H}M$ -algebra whose unit is $e \in \mathbb{Z}e$ and whose multiplication homomorphisms $\mathbb{Z}(x) \otimes \mathbb{Z}(y) \rightarrow \mathbb{Z}(xy)$ are given by $mx \circ ny = (mn)xy$, where mn is multiplication of m and n in the ring \mathbb{Z} .

The augmented case is relevant. A commutative *differential graded augmented algebra* (DGA-algebra) \mathcal{A} over $\mathbb{H}M$ is a commutative DG-algebra over $\mathbb{H}M$ as above equipped with a homomorphism of commutative DG-algebras (the *augmentation*) $\epsilon : \mathcal{A} \rightarrow \mathbb{Z}$. Such an augmentation is entirely determined by its component of degree 0, which is a morphism of $\mathbb{H}M$ -algebras $\epsilon : \mathcal{A}_0 \rightarrow \mathbb{Z}$ such that $\epsilon \partial = 0$. Morphisms of commutative DGA-algebras over $\mathbb{H}M$ are those of commutative DG-algebras which are compatible with the augmentations (i.e., $\epsilon f = \epsilon$).

Remark 2.2. When $M = G$ is a group, the equivalence between the category of abelian groups and the category of $\mathbb{H}G$ -modules, described in Remark 2.1, is symmetric monoidal and, therefore, produces an equivalence between the category of commutative DGA-rings and the category of commutative DGA-algebras over $\mathbb{H}G$. Thus every commutative DGA-ring A defines a *constant* commutative DGA-algebra over $\mathbb{H}G$, equally denoted by A , and each commutative DGA-algebra

over $\mathbb{H}G$, \mathcal{A} , gives rise to the DGA-ring $\mathcal{A}(e)$, which comes with a natural isomorphism of DGA-algebras $\mathcal{A} \cong \mathcal{A}(e)$ whose component at each $x \in G$ is the isomorphism of augmented chain complexes $x_*^{-1} : \mathcal{A}(x) \rightarrow \mathcal{A}(e)$.

3. THE BAR CONSTRUCTION ON COMMUTATIVE DGA-ALGEBRAS OVER $\mathbb{H}M$

Let \mathcal{A} be any given commutative DGA-algebra over $\mathbb{H}M$. As we explain below, \mathcal{A} determines a new commutative DGA-algebra over $\mathbb{H}M$, denoted by $\mathbf{B}(\mathcal{A})$ and called the *bar construction* on \mathcal{A} .

Previously to describe $\mathbf{B}(\mathcal{A})$, let us introduce complexes of $\mathbb{H}M$ -modules $\bar{\mathcal{A}}$, $S\bar{\mathcal{A}}$, and $T^p S\bar{\mathcal{A}}$ for each integer $p \geq 0$, and a double complex of $\mathbb{H}M$ -modules $T^\bullet S\bar{\mathcal{A}}$, as follows:

The *reduced complex* $\bar{\mathcal{A}} = \cdots \rightarrow \bar{\mathcal{A}}_2 \xrightarrow{\partial} \bar{\mathcal{A}}_1 \xrightarrow{\partial} \bar{\mathcal{A}}_0$ is defined to be the cokernel of the unit morphism $\iota : \mathbb{Z} \rightarrow \mathcal{A}$. That is, $\bar{\mathcal{A}} = \cdots \rightarrow \bar{\mathcal{A}}_2 \xrightarrow{\partial} \bar{\mathcal{A}}_1 \xrightarrow{\partial} \bar{\mathcal{A}}_0 = \mathcal{A}_0 / \iota\mathbb{Z}$. Note that ι embeds \mathbb{Z} as a direct summand of the underlying complex \mathcal{A} , since, being $\epsilon : \mathcal{A} \rightarrow \mathbb{Z}$ the augmentation, $\epsilon\iota = id_{\mathbb{Z}}$. We will use below the following notation: For any $x \in M$ and each chain a_x of the chain complex $\mathcal{A}(x)$, $\tilde{\epsilon}(a_x)$ is the integer which express $\epsilon_x(a_x)$ as a multiple of the generator x of the abelian group $\mathbb{Z}(x)$, that is, such that

$$(10) \quad \epsilon_x(a_x) = \tilde{\epsilon}(a_x)x.$$

The complex $S\bar{\mathcal{A}}$ is the *suspension* of $\bar{\mathcal{A}}$, that is, the complex of $\mathbb{H}M$ -modules defined by $(S\bar{\mathcal{A}})_{p+2} = \bar{\mathcal{A}}_{p+1}$, $(S\bar{\mathcal{A}})_1 = \bar{\mathcal{A}}_0 / \iota\mathbb{Z}$, $(S\bar{\mathcal{A}})_0 = 0$, and differential $-\partial$. The *suspension map* is then the morphism of complexes $S : \bar{\mathcal{A}} \rightarrow S\bar{\mathcal{A}}$, of degree 1, defined by

$$S_p = id_{\bar{\mathcal{A}}_p} : \bar{\mathcal{A}}_p \rightarrow (S\bar{\mathcal{A}})_{p+1} = \bar{\mathcal{A}}_p.$$

Note that the sign in the differential of $S\bar{\mathcal{A}}$ is taken so that the equality $\partial S = -S\partial$ holds.

For each $p \geq 1$, let $T^p S\bar{\mathcal{A}}$ be the complex of $\mathbb{H}M$ -modules defined by the iterated tensor product

$$T^p S\bar{\mathcal{A}} = S\bar{\mathcal{A}} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} S\bar{\mathcal{A}} \quad (p \text{ factors}).$$

Thus, for any integer $n \geq 0$ and $x \in M$, the abelian group $(T^p S\bar{\mathcal{A}})_n(x)$ is generated by elements $S\bar{a}_{x_1} \otimes \cdots \otimes S\bar{a}_{x_p}$, that we write as

$$(11) \quad [a_{x_1} | \cdots | a_{x_p}],$$

where the $x_i \in M$ are elements of the monoid such that $x_1 \cdots x_p = x$, and the $a_{x_i} \in \mathcal{A}_{r_i}(x_i)$ are chains of the complexes of abelian groups $\mathcal{A}(x_i)$ whose degrees satisfy that $p + r_1 + \cdots + r_p = n$. On such a generator (11), the differential ∂^\otimes of $T^p S\bar{\mathcal{A}}$ at x ,

$$\partial_x^\otimes : (T^p S\bar{\mathcal{A}})_n(x) \rightarrow (T^p S\bar{\mathcal{A}})_{n-1}(x),$$

acts by

$$\partial_x^\otimes [a_{x_1} | \cdots | a_{x_p}] = - \sum_{i=1}^p (-1)^{e_i-1} [a_{x_1} | \cdots | a_{x_{i-1}} | \partial_{x_i} a_{x_i} | a_{x_{i+1}} | \cdots | a_{x_p}],$$

where the exponents e_i of the signs are $e_0 = 0$ and, for $i \geq 1$,

$$e_i = i + r_1 + \cdots + r_i,$$

and $\partial_{x_i} : \mathcal{A}_{r_i}(x_i) \rightarrow \mathcal{A}_{r_i-1}(x_i)$ is the differential of \mathcal{A} at x_i . Remark that the elements (11) are normalized, in the sense that $[a_{x_1} | \cdots | a_{x_p}] = 0$ whenever some $a_{x_i} = x_{i*}1 \in \mathcal{A}_0(x_i)$.

For $p = 0$, we take $T^0 S\bar{\mathcal{A}}$ to be \mathbb{Z} , but where we write $[]$ for the unit $e \in \mathbb{Z}(e)$. Thus, $T^0 S\bar{\mathcal{A}}$ is the concentrated in degree 0 complex of $\mathbb{H}M$ -modules such that, for any $x \in M$, $T^0 S\bar{\mathcal{A}}(x)$ is the free abelian group on the element $x_*[] (= []$ if $x = e)$, and, for each $x, y \in M$, $y_* : T^0 S\bar{\mathcal{A}}(x) \rightarrow T^0 S\bar{\mathcal{A}}(xy)$ is determined by $y_* x_* [] = (yx)_* []$.

The double complex of $\mathbb{H}M$ -modules

$$\mathbf{T}^\bullet \mathbf{S}\mathcal{A} = \cdots \rightarrow \mathbf{T}^2 \mathbf{S}\bar{\mathcal{A}} \xrightarrow{\partial^\circ} \mathbf{T}^1 \mathbf{S}\bar{\mathcal{A}} \xrightarrow{\partial^\circ} \mathbf{T}^0 \mathbf{S}\bar{\mathcal{A}}$$

is then constructed, thanks to the multiplication \circ in \mathcal{A} , by the morphisms of complexes of $\mathbb{H}M$ -modules $\partial^\circ : \mathbf{T}^p \mathbf{S}\bar{\mathcal{A}} \rightarrow \mathbf{T}^{p-1} \mathbf{S}\bar{\mathcal{A}}$, which are of degree -1 (so that $\partial^\circ \partial^\circ = -\partial^\circ \partial^\circ$) and defined, at any $x \in M$, by the homomorphisms

$$\partial_x^\circ : (\mathbf{T}^p \mathbf{S}\bar{\mathcal{A}})_n(x) \rightarrow (\mathbf{T}^{p-1} \mathbf{S}\bar{\mathcal{A}})_{n-1}(x)$$

given on generators as in (11) by

$$\begin{aligned} \partial_x^\circ[a_{x_1} | \cdots | a_{x_p}] &= \tilde{\epsilon}_{x_1}(a_{x_1}) x_{1*}[a_{x_2} | \cdots | a_{x_p}] \\ &\quad + \sum_{i=1}^{p-1} (-1)^{e_i} [a_{x_1} | \cdots | a_{x_{i-1}} | a_{x_i} \circ a_{x_{i+1}} | a_{x_{i+1}} | \cdots | a_{x_p}] \\ &\quad + (-1)^{e_p} \tilde{\epsilon}_{x_p}(a_{x_p}) x_{p*}[a_{x_1} | \cdots | a_{x_{p-1}}] \end{aligned}$$

(recall the notation $\tilde{\epsilon}$ from (10), and note that the first (resp. last) summand in the above formula is zero whenever the degree r_1 of a_{x_1} in the chain complex $\mathcal{A}(x_1)$ (resp. r_p of a_{x_p}) is higher than zero).

All in all, we are now ready to present the bar construction $\mathbf{B}(\mathcal{A})$. As a graded $\mathbb{H}M$ -module

$$\mathbf{B}(\mathcal{A}) = \cdots \rightarrow \mathbf{B}(\mathcal{A})_2 \xrightarrow{\partial} \mathbf{B}(\mathcal{A})_1 \xrightarrow{\partial} \mathbf{B}(\mathcal{A})_0$$

is defined by the $\mathbb{H}M$ -modules

$$\mathbf{B}(\mathcal{A})_n = \bigoplus_{p \geq 0} (\mathbf{T}^p \mathbf{S}\bar{\mathcal{A}})_n.$$

Notice that $\partial^\circ \mathbf{B}(\mathcal{A})_n \subseteq \mathbf{B}(\mathcal{A})_{n-1}$, $\partial^\circ \mathbf{B}(\mathcal{A})_n \subseteq \mathbf{B}(\mathcal{A})_{n-1}$, and that $(\partial^\circ + \partial^\circ)^2 = 0$. Thus, $\mathbf{B}(\mathcal{A})$ becomes a complex of $\mathbb{H}M$ -modules with differential

$$\partial = \partial^\circ + \partial^\circ : \mathbf{B}(\mathcal{A})_n \rightarrow \mathbf{B}(\mathcal{A})_{n-1}.$$

Proposition 3.1. $\mathbf{B}(\mathcal{A})$ is a commutative DGA-algebra over $\mathbb{H}M$, with multiplication

$$\circ : \mathbf{B}(\mathcal{A}) \otimes_{\mathbb{H}M} \mathbf{B}(\mathcal{A}) \rightarrow \mathbf{B}(\mathcal{A})$$

defined, for integers $m, n \geq 0$ and $x, y \in M$, by the homomorphisms of abelian groups

$$\circ : \mathbf{B}(\mathcal{A})_m(x) \otimes \mathbf{B}(\mathcal{A})_n(y) \rightarrow \mathbf{B}(\mathcal{A})_{m+n}(xy)$$

given by the shuffle products

$$[a_{x_1} | \cdots | a_{x_p}] \circ [a_{x_{p+1}} | \cdots | a_{x_{p+q}}] = \sum_{\sigma} (-1)^{e(\sigma)} [a_{x_{\sigma^{-1}(1)}} | \cdots | a_{x_{\sigma^{-1}(p+q)}}]$$

for any $x_i \in M$ and $a_{x_i} \in \mathcal{A}_{r_i}(x_i)$, $i = 1, \dots, p+q$, such that $x_1 \cdots x_p = x$, $x_{p+1} \cdots x_{p+q} = y$, $p + \sum_{i=1}^p r_i = m$, and $q + \sum_{j=1}^q r_{p+j} = n$, where the sum is taken over all (p, q) -shuffles σ and, for each σ , the exponent of the sign is $e(\sigma) = \sum (1+r_i)(1+r_{p+j})$ summed over all pairs $(i, p+j)$ such that $\sigma(i) > \sigma(p+j)$.

The unit is $[] \in \mathbf{B}(\mathcal{A})_0(e)$, that is, the unit morphism $\iota : \mathbb{Z} \rightarrow \mathbf{B}(\mathcal{A})$ is the isomorphism of $\mathbb{H}M$ -modules $\iota : \mathbb{Z} \cong \mathbf{B}(\mathcal{A})_0$ given by $\iota_x(x) = x_*[]$, for any $x \in M$, and the augmentation $\epsilon : \mathbf{B}(\mathcal{A}) \rightarrow \mathbb{Z}$ is defined by the isomorphism of $\mathbb{H}M$ -modules $\epsilon = \iota^{-1} : \mathbf{B}(\mathcal{A})_0 \cong \mathbb{Z}$ such that $\epsilon_x(x_*[]) = x$, for any $x \in M$.

Proof. We give an indirect proof, by using that the category of $\mathbb{H}M$ -modules is closely related to the category $\mathbb{Z}M$ -Mod, of ordinary modules over the monoid ring $\mathbb{Z}M$.

There is an exact faithful functor $\Gamma : \mathbb{H}M\text{-Mod} \rightarrow \mathbb{Z}M\text{-Mod}$, which carries any $\mathbb{H}M$ -module \mathcal{A} to the $\mathbb{Z}M$ -module defined by the abelian group $\Gamma\mathcal{A} = \bigoplus_{x \in M} \mathcal{A}(x)$, with M -action of an

element $y \in M$ on an element $a_x \in \mathcal{A}(x)$ given by $y a_x = y_* a_x \in \mathcal{A}(xy)$. This functor Γ is left adjoint to the functor which associates to any $\mathbb{Z}M$ -module A the constant on objects $\mathbb{H}M$ -module defined by the underlying abelian group A , with $y_* : A \rightarrow A$, for any $y \in M$, the homomorphism of multiplication by y [15].

It is plain to see that Γ is a symmetric strict monoidal functor, that is, $\Gamma\mathbb{Z} = \mathbb{Z}M$, for any $\mathbb{H}M$ -modules \mathcal{A} and \mathcal{B} , $\Gamma(\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) = \Gamma\mathcal{A} \otimes_{\mathbb{Z}M} \Gamma\mathcal{B}$, and it carries the associativity, unit, and commutativity constraints of the monoidal category of $\mathbb{H}M$ -modules to the corresponding ones of the category of $\mathbb{Z}M$ -modules. Then, the same properties hold for the induced functor Γ from the symmetric monoidal category of complexes of $\mathbb{H}M$ -modules to the symmetric monoidal category of complexes of $\mathbb{Z}M$ -modules. It follows that Γ transform commutative monoids in the category of complexes of $\mathbb{H}M$ -modules (i.e. commutative DG-algebras over $\mathbb{H}M$) to commutative monoids in the category of $\mathbb{Z}M$ -modules (i.e., commutative DG-algebras over $\mathbb{Z}M$), and therefore Γ also transform commutative DGA-algebras over $\mathbb{H}M$ to commutative DGA-algebras over the monoid ring $\mathbb{Z}M$.

Now, given \mathcal{A} , a commutative DGA-algebra over $\mathbb{H}M$, let $\mathbf{B}(\Gamma\mathcal{A})$ be the commutative DGA-algebra over $\mathbb{Z}M$ obtained by applying the ordinary Eilenberg-Mac Lane bar construction on $\Gamma\mathcal{A}$ [19, Chapter X, Theorem 12.1]. A direct comparison shows that $\mathbf{B}(\Gamma\mathcal{A}) = \Gamma\mathbf{B}(\mathcal{A})$ as complexes of $\mathbb{Z}M$ -modules, and also that its multiplication, unit, and augmentation are, respectively, just the morphisms

$$\begin{aligned} \mathbf{B}(\Gamma\mathcal{A}) \otimes_{\mathbb{Z}M} \mathbf{B}(\Gamma\mathcal{A}) &= \Gamma(\mathbf{B}(\mathcal{A}) \otimes_{\mathbb{H}M} \mathbf{B}(\mathcal{A})) \xrightarrow{\Gamma\circ} \Gamma\mathbf{B}(\mathcal{A}) = \mathbf{B}(\Gamma\mathcal{A}), \\ \mathbb{Z}M = \Gamma\mathbb{Z} &\xrightarrow{\Gamma\iota} \Gamma\mathbf{B}(\mathcal{A}) = \mathbf{B}(\Gamma\mathcal{A}), \quad \mathbf{B}(\Gamma\mathcal{A}) = \Gamma\mathbf{B}(\mathcal{A}) \xrightarrow{\Gamma\epsilon} \Gamma\mathbb{Z} = \mathbb{Z}M. \end{aligned}$$

Then, as $\mathbf{B}(\Gamma\mathcal{A})$ is actually a commutative DGA-algebra over $\mathbb{Z}M$, it follows that the equalities

$$\begin{aligned} \Gamma(\circ \circ (\circ \otimes id_{\mathbf{B}(\mathcal{A})})) &= \Gamma(\circ (id_{\mathbf{B}(\mathcal{A})} \otimes \circ)), \quad \Gamma(\circ c_{\mathbf{B}(\mathcal{A}), \mathbf{B}(\mathcal{A})}) = \Gamma\circ, \\ \Gamma(\circ(\iota \otimes id_{\mathbf{B}(\mathcal{A})})) &= \Gamma\iota_{\mathbf{B}(\mathcal{A})}, \quad \Gamma(\circ(\epsilon \otimes \epsilon)) = \Gamma(\epsilon\circ), \quad \Gamma(\epsilon\iota) = \Gamma id_{\mathbb{Z}}. \end{aligned}$$

hold. Therefore, the result, that is, that $\mathbf{B}(\mathcal{A})$ is a commutative DGA-algebra over $\mathbb{H}M$, follows since the functor Γ is faithful. \square

Remark 3.2. Observe, as in [7, §7], that the shuffle product \circ on $\mathbf{B}(\mathcal{A})$ can also be expressed by the recursive formula below, where $\alpha = [a_{x_1} | \cdots | a_{x_p}] \in \mathbf{B}(\mathcal{A})_r(x)$, $\beta = [b_{y_1} | \cdots | b_{y_q}] \in \mathbf{B}(\mathcal{A})_s(y)$, $a_z \in \mathcal{A}_m(z)$ and $b_t \in \mathcal{A}_n(t)$.

$$(12) \quad [\alpha | a_z] \circ [\beta | b_t] = [[\alpha | a_z] \circ \beta | b_t] + (-1)^{r(n+s+1)}[\alpha \circ [\beta | b_t] | a_z]$$

Let us stress the *suspension* morphism of complexes of $\mathbb{H}M$ -modules, of degree 1 (hence satisfying $\partial S = -S\partial$),

$$(13) \quad S : \mathcal{A} \rightarrow \mathbf{B}(\mathcal{A}),$$

which is defined, at any $x \in M$, by $S_x a_x = [a_x] \in \mathbf{B}(\mathcal{A})(x)$, for any chain a_x of $\mathcal{A}(x)$.

Such as Mac Lane did in [19, Chapter X, §12] for ordinary commutative DGA-algebras over a commutative ring, the cohomology of a commutative DGA-algebra over $\mathbb{H}M$ can be defined in “stages” or “levels”. If \mathcal{A} is any commutative DGA-algebra over $\mathbb{H}M$, then $\mathbf{B}(\mathcal{A})$ is again a commutative DGA-algebra over $\mathbb{H}M$, so an iteration is possible to form $\mathbf{B}^r(\mathcal{A})$ for each integer $r \geq 1$. Hence, we define the *rth level cohomology groups of \mathcal{A} with coefficients in a $\mathbb{H}M$ -module \mathcal{B}* , denoted by $H^n(\mathcal{A}, r; \mathcal{B})$, as

$$H^n(\mathcal{A}, r; \mathcal{B}) = H^n(\text{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}), \mathcal{B})), \quad n = 0, 1, \dots,$$

where $\text{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}), \mathcal{B})$ is the cochain complex obtained by applying the functor $\text{Hom}_{\mathbb{H}M}(-, \mathcal{B})$ to the underlying chain complex of $\mathbb{H}M$ -modules $\mathbf{B}^r(\mathcal{A})$.

Remark 3.3. When the bar construction above is applied on the constant DGA-algebra over $\mathbb{H}M$ defined by a commutative DGA-ring A , the result is just the constant DGA-algebra over $\mathbb{H}M$ defined by the commutative DGA-ring obtained by applying on A the Eilenberg-Mac Lane reduced bar construction. Hence, the notation $\mathbf{B}(A)$ is not confusing.

If \mathcal{A} and \mathcal{B} are commutative DGA-algebras over $\mathbb{H}M$, then we say that two morphisms of DGA-algebras $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{A}$ form a *contraction* whenever $fg = id_{\mathcal{B}}$, and there exists a homotopy of morphisms of complexes $\Phi : gf \Rightarrow id_{\mathcal{A}}$ satisfying the conditions

$$(14) \quad \Phi g = 0, \quad f \Phi = 0, \quad \Phi \Phi = 0.$$

Paralleling the proof by Eilenberg and MacLane of [7, Theorem 12.1], one proves the following:

Lemma 3.4. *If $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{A}$ form a contraction of commutative DGA-algebras over $\mathbb{H}M$, then the induced $\mathbf{B}(f) : \mathbf{B}(\mathcal{A}) \rightarrow \mathbf{B}(\mathcal{B})$ and $\mathbf{B}(g) : \mathbf{B}(\mathcal{B}) \rightarrow \mathbf{B}(\mathcal{A})$ also form a contraction.*

4. FREE $\mathbb{H}M$ -MODULES

Let $\mathbf{Set}_{\downarrow M}$ be the comma category of sets over the underlying set of M ; that is, the category whose objects $S = (S, \pi)$ are sets S endowed with a map $\pi : S \rightarrow M$, and whose morphisms are maps $\varphi : S \rightarrow T$ such that $\pi \varphi = \pi$. There is a *forgetful functor*

$$\mathcal{U} : \mathbb{H}M\text{-Mod} \rightarrow \mathbf{Set}_{\downarrow M},$$

which carries any $\mathbb{H}M$ -module \mathcal{A} to the disjoint union set

$$\mathcal{U}\mathcal{A} = \bigcup_{x \in M} \mathcal{A}(x) = \{(x, a_x) \mid x \in M, a_x \in \mathcal{A}(x)\},$$

endowed with the projection map $\pi : \mathcal{U}\mathcal{A} \rightarrow M$, $\pi(x, a_x) = x$. A morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is sent to the map $\mathcal{U}f : \mathcal{U}\mathcal{A} \rightarrow \mathcal{U}\mathcal{B}$ given by $\mathcal{U}f(x, a_x) = (x, f_x a_x)$. There is also a *free $\mathbb{H}M$ -module functor*

$$(15) \quad \mathcal{Z} : \mathbf{Set}_{\downarrow M} \rightarrow \mathbb{H}M\text{-Mod},$$

which is defined as follows: If S is any set over M , then $\mathcal{Z}S$ is the $\mathbb{H}M$ -module such that, for each $x \in M$,

$$\mathcal{Z}S(x) = \mathbb{Z}\{(u, s) \in M \times S \mid u \pi(s) = x\}$$

is the free abelian group with generators all pairs (u, s) , where $u \in M$ and $s \in S$, such that $u \pi(s) = x$. We usually write (e, s) simply by s ; so that each element of $s \in S$ is regarded as an element $s \in \mathcal{Z}S(\pi s)$. For any $x, y \in M$, the homomorphism

$$y_* : \mathcal{Z}S(x) \rightarrow \mathcal{Z}S(xy)$$

is defined on generators by $y_*(u, s) = (uy, s)$. If $\varphi : S \rightarrow T$ is any map of sets over M , the induced morphism $\mathcal{Z}\varphi : \mathcal{Z}S \rightarrow \mathcal{Z}T$ is given, at each $x \in M$, by the homomorphism $(\mathcal{Z}\varphi)_x : \mathcal{Z}S(x) \rightarrow \mathcal{Z}T(x)$ defined on generators by $(\mathcal{Z}\varphi)_x(u, s) = (u, \varphi s)$.

Proposition 4.1. *The functor \mathcal{Z} is left adjoint to the functor \mathcal{U} . Thus, for S any set over M , to each $\mathbb{H}M$ -module \mathcal{A} and each list of elements $a_s \in \mathcal{A}(\pi s)$, one for each $s \in S$, there is a unique morphism of $\mathbb{H}M$ -modules $f : \mathcal{Z}S \rightarrow \mathcal{A}$ with $f_{\pi s}(s) = a_s$ for every $s \in S$.*

Proof. At any set S over M , the unit of the adjunction is the map

$$\nu : S \rightarrow \mathcal{U}\mathcal{Z}S = \{(x, a_x) \mid x \in M, a_x \in \mathcal{Z}S(x)\}, \quad s \mapsto (\pi s, s).$$

If \mathcal{A} is a $\mathbb{H}M$ -module and $\varphi : S \rightarrow \mathcal{U}\mathcal{A}$ is any map over M , then, the unique morphism of $\mathbb{H}M$ -modules $f : \mathcal{Z}S \rightarrow \mathcal{A}$ such that $(\mathcal{U}f)\nu = \varphi$ is determined by the equations $f_x(u, s) = u_*\varphi(s)$, for any $x \in M$ and $(u, s) \in M \times S$ with $u \pi(s) = x$. \square

The category $\mathbf{Set} \downarrow_M$ has a symmetric monoidal structure, where the tensor product of two sets over M , say S and T , is the cartesian product set of $S \times T$ with $\pi(s, t) = \pi(s)\pi(t)$. The unit object is provided by the unitary set $\{e\}$ with $\pi(e) = e \in M$, and the associativity, unit, and commutativity constraints are the obvious ones. Hereafter, the category $\mathbf{Set} \downarrow_M$ will be considered with this monoidal structure¹.

Proposition 4.2. *The free $\mathbb{H}M$ -module functor (15) is symmetric monoidal, that is, there are natural and coherent isomorphisms of $\mathbb{H}M$ -modules*

$$\mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T, \quad \mathcal{Z}\{e\} \cong \mathbb{Z},$$

for S and T any sets over M .

Proof. For S, T any given sets over M , the isomorphism $f : \mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T$ is the morphism of $\mathbb{H}M$ -modules such that, for any $(s, t) \in S \times T$, $f_{\pi(s, t)}(s, t) = s \otimes t$. Observe that, for any $x \in M$, the abelian group $\mathcal{Z}(S \times T)(x)$ is free with generators the elements $(u, s, t) = u_*(s, t)$, with $u \in M$, $s \in S$, and $t \in T$, such that $u\pi(s)\pi(t) = x$, while $(\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$ is the abelian group generated by the elements $(u, s) \otimes (v, t) = u_*s \otimes v_*t$, with $u, v \in M$, $s \in S$, and $t \in T$, such that $u\pi(s)v\pi(t) = x$, with the relations $u_*s \otimes v_*t = (uv)_*(s \otimes t)$. Then, the homomorphism $f_x : \mathcal{Z}(S \times T)(x) \rightarrow (\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$, which acts on elements of the basis by $f_x(u_*(s, t)) = u_*(s \otimes t)$, is clearly an isomorphism of abelian groups.

The isomorphism $f : \mathcal{Z}\{e\} \cong \mathbb{Z}$ is the morphism of $\mathbb{H}M$ -modules such that $f_e(e) = e$. Observe that, for any $x \in M$, the isomorphism f_x is the composite

$$\mathcal{Z}\{e\}(x) = \mathbb{Z}\{(u, e) \mid ue = x\} = \mathcal{Z}\{(x, e)\} \cong \mathbb{Z}\{x\} = \mathbb{Z}(x).$$

It is straightforward to see that the isomorphisms f above are natural and coherent, so that \mathcal{Z} is actually a symmetric monoidal functor. \square

Corollary 4.3. *For S and T any two sets over M , the tensor product $\mathbb{H}M$ -module $\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T$ is free on the set of elements $s \otimes t$, $s \in S$, $t \in T$, with $\pi(s \otimes t) = \pi(s)\pi(t)$.*

Since the functor \mathcal{Z} is symmetric monoidal, it transports commutative monoids in $\mathbf{Set} \downarrow_M$ to commutative monoids in $\mathbb{H}M\text{-Mod}$, that is, to algebras over $\mathbb{H}M$. As a commutative monoid in the symmetric monoidal category $\mathbf{Set} \downarrow_M$ is merely a commutative monoid over M , that is, a commutative monoid S endowed with a homomorphism $\pi : S \rightarrow M$, the corollary below follows.

Corollary 4.4. *If S is a commutative monoid over M , then the free $\mathbb{H}M$ -module $\mathcal{Z}S$ is an algebra over $\mathbb{H}M$. The multiplication morphism $\circ : \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}S \rightarrow \mathcal{Z}S$ is the composite*

$$\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}S \cong \mathcal{Z}(S \times S) \xrightarrow{\mathcal{Z}m} \mathcal{Z}S,$$

where $m : S \times S \rightarrow S$ is the homomorphism of multiplication in S , $m(s, s') = ss'$, and the unit morphism $\iota : \mathbb{Z} \rightarrow \mathcal{Z}S$ is the composite $\mathbb{Z} \cong \mathcal{Z}\{e\} \xrightarrow{\mathcal{Z}i} \mathcal{Z}S$, where $i : \{e\} \rightarrow S$ is the trivial homomorphism mapping the unit of M to the unit of S .

5. THE COHOMOLOGY GROUPS $H^n(M, r; \mathcal{A})$

Let us consider the commutative monoid M over itself with $\pi = id_M : M \rightarrow M$. Then, by Corollary 4.4, the free $\mathbb{H}M$ -module $\mathcal{Z}M$ is an algebra over $\mathbb{H}M$. Explicitly, this is described as follows: For each $x \in M$,

$$\mathcal{Z}M(x) = \mathbb{Z}\{(u, v) \mid uv = x\}$$

¹The category $\mathbf{Set} \downarrow_M$ has a different monoidal structure where the tensor product is given by the fibre-product $S \times_M T$ with $\pi(s, t) = \pi(s) = \pi(t)$.

is the free abelian group with generators all pairs $(u, v) \in M \times M$ such that $uv = x$. For any $x, y \in M$, the homomorphism $y_* : \mathcal{Z}M(x) \rightarrow \mathcal{Z}M(xy)$ is given on generators by $y_*(u, v) = (yu, v)$, and the homomorphism of multiplication

$$\circ : \mathcal{Z}M(x) \otimes \mathcal{Z}M(y) \rightarrow \mathcal{Z}M(xy)$$

is defined on generators by $(u, v) \otimes (w, t) \mapsto (u, v) \circ (w, t) = (uw, vt)$, for any $u, v, w, t \in M$ such that $uv = x$ and $wt = y$. The unit is $(e, e) \in \mathcal{Z}M(e)$. We see each element $x \in M$ as an element of $\mathcal{Z}M(x)$ by means of the identification $x = (e, x)$, so that that any generator (u, v) of $\mathcal{Z}M(x)$ can be write as u_*v .

By Proposition 4.1, if \mathcal{A} is any $\mathbb{H}M$ -module, for any list of elements $a_x \in \mathcal{A}(x)$, one for each $x \in M$, there is an unique morphism of $\mathbb{H}M$ -modules $f : \mathcal{Z}M \rightarrow \mathcal{A}$ such that each homomorphism $f_x : \mathcal{Z}M(x) \rightarrow \mathcal{A}(x)$ verifies that $f_x(x) = a_x$ (explicitly, f_x acts on generators by $f_x(u, v) = u_*a_v$). Furthermore, it is plain to see that, if \mathcal{A} is an algebra over $\mathbb{H}M$, then f is a morphism of algebras if and only if $a_e = 1$ and $a_x \circ a_y = a_{xy}$ for all $x, y \in M$.

Hereafter, we regard $\mathcal{Z}M$ as a commutative DGA-algebra over $\mathbb{H}M$ with the trivial grading, that is, with $(\mathcal{Z}M)_n = 0$ for $n > 0$ and $(\mathcal{Z}M)_0 = \mathcal{Z}M$, and with augmentation the morphism of $\mathbb{H}M$ -algebras

$$\epsilon : \mathcal{Z}M \rightarrow \mathbb{Z},$$

such that, for any $x \in M$, $\epsilon_x(x) = x \in \mathbb{Z}(x)$. Then, we define, for each integer $r \geq 1$, the r th level cohomology groups of the commutative monoid M with coefficients in a $\mathbb{H}M$ -module \mathcal{A} by

$$(16) \quad H^n(M, r; \mathcal{A}) = H^n(\mathcal{Z}M, r; \mathcal{A}), \quad n = 0, 1, \dots,$$

or, in other words,

$$H^n(M, r; \mathcal{A}) = H^n(\text{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{Z}M), \mathcal{A})),$$

where $\text{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{Z}M), \mathcal{A})$ is the cochain complex obtained by applying the abelian group valued functor $\text{Hom}_{\mathbb{H}M}(-, \mathcal{A})$ to the neglected chain complex of $\mathbb{H}M$ -modules $\mathbf{B}^r(\mathcal{Z}M)$.

Remark 5.1. When $M = G$ is an abelian group, $\mathcal{Z}G$ is isomorphic to the constant DGA-algebra over $\mathbb{H}G$ defined by the commutative DGA-ring $\mathcal{Z}G(e)$ (see Remark 2.2), which is itself isomorphic to the trivially graded DGA-ring defined by the group ring $\mathbb{Z}G$ with augmentation the ring homomorphism $\alpha : \mathbb{Z}G \rightarrow \mathbb{Z}$ such that $\alpha(x) = 1$ for any $x \in G$. To see this, observe that $\mathcal{Z}G(e)$ is the commutative ring whose underlying abelian group is freely generated by the elements of the form (x^{-1}, x) , $x \in G$, with multiplication such that $(x^{-1}, x) \circ (y^{-1}, y) = ((xy)^{-1}, xy)$, and unit $(e, e) = e$. The map $(x^{-1}, x) \mapsto x$ clearly determines a ring isomorphism between $\mathcal{Z}G(e)$ and the group ring $\mathbb{Z}G$, which is compatible with the corresponding augmentations.

Hence, for any integer $r \geq 1$, $\mathbf{B}^r(\mathcal{Z}G) \cong \mathbf{B}^r(\mathbb{Z}G)$ (see Remark 3.3)², and therefore for any abelian group A , regarded as a constant $\mathbb{H}G$ -module, there are natural isomorphisms

$$\text{Hom}_{\mathbb{H}G}(\mathbf{B}^r(\mathcal{Z}G), A) \cong \text{Hom}_{\mathbb{H}G}(\mathbf{B}^r(\mathbb{Z}G), A) \cong \text{Hom}(\mathbf{B}^r(\mathbb{Z}G), A)$$

showing that the r th level cohomology groups $H^n(G, r; A)$ in (16) agree with those by Eilenberg and Mac Lane in [7], which compute the cohomology of the spaces $K(G, r)$ by means of natural isomorphisms $H^n(K(G, r), A) \cong H^n(G, r; A)$.

From here on, this section is dedicated to show explicit cochain descriptions for some of these cohomology groups, starting with those of first level

$$H^n(M, 1; \mathcal{A}) = H^n(\text{Hom}_{\mathbb{H}M}(\mathbf{B}(\mathcal{Z}M), \mathcal{A})).$$

Let us analyze the underlying complex $\mathbf{B}(\mathcal{Z}M)$. For any integer $n \geq 1$,

$$\mathbf{B}(\mathcal{Z}M)_n = \overline{\mathcal{Z}M} \otimes_{\mathbb{H}M} \overset{(n \text{ factors})}{\cdots} \otimes_{\mathbb{H}M} \overline{\mathcal{Z}M},$$

²The commutative DGA-rings $\mathbf{B}^r(\mathbb{Z}G)$ are denoted by $A_N(G, r)$ in [7]

where $\overline{\mathcal{Z}M} = \mathcal{Z}M/\iota\mathbb{Z} = \mathcal{Z}M/\mathcal{Z}\{e\} \cong \mathcal{Z}M^*$ is a free $\mathbb{H}M$ -module on $M^* = M \setminus \{e\}$ with $\pi : M^* \rightarrow M$ the inclusion map. Then, by construction and Proposition 4.2, we have that

- The $\mathbb{H}M$ -module $\mathbf{B}(\mathcal{Z}M)_0$ is free on the unitary set $\{[\]\}$ with $\pi[\] = e$ and, for any $n \geq 1$, $\mathbf{B}(\mathcal{Z}M)_n$ is a free $\mathbb{H}M$ -module generated by the set over M consisting of n -tuples of elements of M

$$\alpha_n = [x_1 | \cdots | x_n], \quad \text{with } \pi\alpha_n = x_1 \cdots x_n,$$

which we call generic n -cells of $\mathbf{B}(\mathcal{Z}M)$, with the relations $\alpha_n = 0$ whenever some $x_i = e$.

- The differential $\partial : \mathbf{B}(\mathcal{Z}M)_n \rightarrow \mathbf{B}(\mathcal{Z}M)_{n-1}$ is the morphism of $\mathbb{H}M$ -modules such that, for each $x \in M$ and any generic n -cell $[x_1 | \cdots | x_n]$ with $x_1 \cdots x_n = x$,

$$\begin{aligned} \partial_x [x_1 | \cdots | x_n] &= x_{1*} [x_2 | \cdots | x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1 | \cdots | x_i x_{i+1} | \cdots | x_n] \\ &\quad + (-1)^n x_{n*} [x_1 | \cdots | x_{n-1}]. \end{aligned}$$

Hence, Proposition 4.1 gives the following.

Theorem 5.2. *For any $\mathbb{H}M$ -module \mathcal{A} , the cohomology groups $H^n(M, 1; \mathcal{A})$ can be computed as the cohomology groups of the cochain complex of normalized 1st level cochains of M with values in \mathcal{A} ,*

$$(17) \quad C(M, 1; \mathcal{A}) : 0 \rightarrow C^0(M, 1; \mathcal{A}) \xrightarrow{\partial^0} C^1(M, 1; \mathcal{A}) \xrightarrow{\partial^1} C^2(M, 1; \mathcal{A}) \xrightarrow{\partial^2} \cdots,$$

where

- $C^0(M, 1; \mathcal{A}) = \mathcal{A}(e)$, and for $n \geq 1$, $C^n(M, 1; \mathcal{A})$ is the abelian group, under pointwise addition, of functions

$$f : M^n \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$$

such that $f(x_1, \dots, x_n) \in \mathcal{A}(x_1 \cdots x_n)$ and $f(x_1, \dots, x_n) = 0$ whenever some $x_i = e$,

- $\partial^0 = 0$, and for $n \geq 1$, the coboundary $\partial^n : C^n(M, 1; \mathcal{A}) \rightarrow C^{n+1}(M, 1; \mathcal{A})$ is given by

$$\begin{aligned} (\partial^n f)(x_1, \dots, x_{n+1}) &= x_{1*} f(x_2, \dots, x_{n+1}) + \sum_{i=1}^n (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} x_{n+1*} f(x_1, \dots, x_n). \end{aligned}$$

Let us now recall that the so-called Leech cohomology groups [16] of a (not necessarily commutative) monoid M , which we denote here by $H^n(M, \mathcal{A})$, take coefficients in $\mathbb{D}M$ -modules, that is, in abelian group valued functors on the category $\mathbb{D}M$, whose set of objects is M and set of arrows $M \times M \times M$, where $(x, y, z) : y \rightarrow xyz$. Composition is given by $(u, xyz, v)(x, y, z) = (ux, y, zv)$, and the identity morphism of any object x is $(e, x, e) : x \rightarrow x$. For any $\mathbb{D}M$ -module $\mathcal{A} : \mathbb{D}M \rightarrow \mathbf{Ab}$, if we write $\mathcal{A}(x, y, z) = x_* z^* : \mathcal{A}(y) \rightarrow \mathcal{A}(xyz)$, then we see that \mathcal{A} consists of abelian groups $\mathcal{A}(x)$, one for each $x \in M$, and homomorphisms

$$x_* : \mathcal{A}(y) \rightarrow \mathcal{A}(xy), \quad x^* : \mathcal{A}(y) \rightarrow \mathcal{A}(yx),$$

for each $x, y \in M$, such that the equations below hold.

$$\begin{aligned} x_* y_* &= (xy)_* : \mathcal{A}(z) \rightarrow \mathcal{A}(xyz), & y^* x^* &= (xy)^* : \mathcal{A}(z) \rightarrow \mathcal{A}(zxy), \\ e_* &= e^* = id_{\mathcal{A}(x)} : \mathcal{A}(x) \rightarrow \mathcal{A}(x), & x_* y^* &= y^* x_* : \mathcal{A}(z) \rightarrow \mathcal{A}(xzy). \end{aligned}$$

When the monoid M is commutative, as it is in our case, there is a full functor $\mathbb{D}M \rightarrow \mathbb{H}M$, which is the identity on objects and carries a morphism $(x, y, z) : y \rightarrow xyz$ of $\mathbb{D}M$ to the morphism $(y, xz) : y \rightarrow xyz$ of $\mathbb{H}M$. Composition with this functor induces a full embedding of $\mathbb{H}M$ -Mod into $\mathbb{D}M$ -mod, whose image consists of the *symmetric $\mathbb{D}M$ -modules*, that is, those

satisfying that $x_* = x^* : \mathcal{A}(y) \rightarrow \mathcal{A}(xy)$, for all $x, y \in M$ [16, Chapter II, 7.15]. Thus, $\mathbb{H}M$ -modules and symmetric $\mathbb{D}M$ -modules are the same thing.

As a direct inspection shows that, for any $\mathbb{H}M$ -module \mathcal{A} , the cochain complex $C(M, 1; \mathcal{A})$ in (17) coincides with the *standard normalized cochain complex* of M with coefficients in \mathcal{A} by Leech [16, Chapter II, 2.12], next theorem follows.

Proposition 5.3. *For any $\mathbb{H}M$ -module \mathcal{A} , there are natural isomorphisms*

$$H^n(M, 1; \mathcal{A}) \cong H_L^n(M, \mathcal{A}), \quad n = 0, 1, \dots$$

We now analyze the complex of $\mathbb{H}M$ -modules $\mathbf{B}^r(\mathcal{Z}M)$ for $r \geq 2$ any integer. By construction,

- $\mathbf{B}^r(\mathcal{Z}M)_0$ is the free $\mathbb{H}M$ -module on the unitary set consisting of the 0-tuple

$$[\], \quad \text{with } \pi[\] = e,$$

which we call the generic 0-cell of $\mathbf{B}^r(\mathcal{Z}M)$,

and, for $n \geq 1$,

$$\mathbf{B}^r(\mathcal{Z}M)_n = \bigoplus_{p + \sum n_i = n} \overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_1} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} \overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_p}.$$

Since $\overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_0 = 0$ while, for $n_i \geq 1$, $\overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_i} = \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_i}$, it follows by induction on r that

- $\mathbf{B}^r(\mathcal{Z}M)_n = 0$ for $0 < n < r$,

and that, for any $r \leq n$,

$$\mathbf{B}^r(\mathcal{Z}M)_n = \bigoplus_{\substack{n_1, \dots, n_p \geq r-1 \\ p + \sum n_i = n}} \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_1} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_p}.$$

Then, if we denote by $|_r$ the symbol $|$ used for the tensor product in the construction of $\mathbf{B}^r(\mathcal{Z}M)$ from $\mathbf{B}^{r-1}(\mathcal{Z}M)$, by Proposition 4.2 and induction, we see that

- $\mathbf{B}^r(\mathcal{Z}M)_n$, for $r \leq n$, is a free $\mathbb{H}M$ -module generated by the set over M consisting of all p -tuples, which we call generic n -cells of $\mathbf{B}^r(\mathcal{Z}M)$,

$$\alpha_n = [\alpha_{n_1}|_r \alpha_{n_2}|_r \cdots |_r \alpha_{n_p}], \quad \text{with } \pi \alpha_n = \pi \alpha_{n_1} \cdots \pi \alpha_{n_p},$$

of generic n_i -cells of $\mathbf{B}^{r-1}(\mathcal{Z}M)$, such that $n_i \geq r-1$ and $p + \sum n_i = n$, with the relations $\alpha_n = 0$ whenever some $\alpha_{n_i} = 0$.

Let us stress that a generic n -cell α_n of any $\mathbf{B}^r(\mathcal{Z}M)$ is actually a generator of the abelian group $\mathbf{B}^r(\mathcal{Z}M)_n(\pi \alpha_n)$. Indeed, for each $x \in M$, $\mathbf{B}^r(\mathcal{Z}M)_n(x)$ is the free abelian group generated by the elements $u_* \alpha_n$ with u an element of M and the α_n any non-zero generic n -cell of $\mathbf{B}^r(\mathcal{Z}M)$ such that $u \pi \alpha_n = x$. Arbitrary elements of the groups $\mathbf{B}^r(\mathcal{Z}M)_n(x)$, are referred as n -chains of $\mathbf{B}^r(\mathcal{Z}M)$.

For any $r \geq 1$, the multiplication \circ_r of $\mathbf{B}^r(\mathcal{Z}M)$ is given by the morphism of $\mathbb{H}M$ -modules

$$\circ_r : \mathbf{B}^r(\mathcal{Z}M)_n \otimes_{\mathbb{H}M} \mathbf{B}^r(\mathcal{Z}M)_m \rightarrow \mathbf{B}^r(\mathcal{Z}M)_{n+m}$$

which, according to Proposition 4.1, are determined on generic cells by the shuffle product

$$[\alpha_{n_1}|_r \cdots |_r \alpha_{n_p}] \circ_r [\alpha_{n_{p+1}}|_r \cdots |_r \alpha_{n_{p+q}}] = \sum_{\sigma} (-1)^{e(\sigma)} [\alpha_{n_{\sigma^{-1}(1)}}|_r \cdots |_r \alpha_{n_{\sigma^{-1}(p+q)}}],$$

where the sum is taken over all (p, q) -shuffles σ and $e(\sigma) = \sum (1 + n_i)(1 + n_{p+j})$ summed over all pairs $(i, p+j)$ such that $\sigma(i) > \sigma(p+j)$. In particular, for $r = 1$,

$$(18) \quad [x_1 | \cdots | x_n] \circ_1 [x_{n+1} | \cdots | x_{n+m}] = \sum_{\sigma} (-1)^{e(\sigma)} [x_{\sigma^{-1}(1)} | \cdots | x_{\sigma^{-1}(n+m)}],$$

where the sum is taken over all (n, m) -shuffles σ and $e(\sigma)$ is the sign of the shuffle.

Then, for $r \geq 2$,

- the boundary $\partial : \mathbf{B}^r(\mathcal{Z}M)_n \rightarrow \mathbf{B}^r(\mathcal{Z}M)_{n-1}$ is the morphism of $\mathbb{H}M$ -modules recursively defined, on any generic n -cell $\alpha_n = [\alpha_{n_1}|_r \cdots |_r \alpha_{n_p}]$ of $\mathbf{B}^r(\mathcal{Z}M)$ with $\pi\alpha_n = x$ and $\pi\alpha_{n_i} = x_i$, by

$$\begin{aligned} \partial_x \alpha_n = & - \sum_{i=1}^p (-1)^{e_{i-1}} [\alpha_{n_1}|_r \cdots |_r \alpha_{n_{i-1}}|_r \partial_{x_i} \alpha_{n_i}|_r \alpha_{n_{i+1}}|_r \cdots |_r \alpha_{n_p}] \\ & + \sum_{i=1}^{p-1} (-1)^{e_i} [\alpha_{n_1}|_r \cdots |_r \alpha_{n_{i-1}}|_r \alpha_{n_i} \circ_{r-1} \alpha_{n_{i+1}}|_r \alpha_{n_{i+2}}|_r \cdots |_r \alpha_{n_p}], \end{aligned}$$

where the exponents e_i of the signs are $e_i = i + \sum n_i$.

In the above formula, the term $\partial_{x_i} \alpha_{n_i}$, which refers to the differential of α_{n_i} in $\mathbf{B}^{r-1}(\mathcal{Z}M)$, or $\alpha_{n_i} \circ_{r-1} \alpha_{n_{i+1}}$, is not in general a generic cell of $\mathbf{B}^{r-1}(\mathcal{Z}M)$ but a chain; the term is to be expanded by linearity.

Recall now that we have the embedding suspensions (13), $S : \mathbf{B}^{r-1}(\mathcal{Z}M) \hookrightarrow \mathbf{B}^r(\mathcal{Z}M)$, through which we identify any generic $(n-1)$ -cell α_{n-1} of $\mathbf{B}^{r-1}(\mathcal{Z}M)$ with the generic n -cell $S\alpha_{n-1} = [\alpha_{n-1}]$ of $\mathbf{B}^r(\mathcal{Z}M)$. Hence, by induction, one proves that any generic n -cell of any $\mathbf{B}^r(\mathcal{Z}M)$ can be uniquely written in the form

$$\alpha_n = [x_1|_{k_1} x_2|_{k_2} \cdots |_{k_{m-1}} x_m]$$

with $x_i \in M$, $1 \leq m$, $1 \leq k_i \leq r$, and $r + \sum_{i=1}^{m-1} k_i = n$. So written, we have $\pi\alpha_n = x_1 \cdots x_m$, and $\alpha_n = 0$ if $x_i = e$ for some i . Observe that if some $k_i = r$, then $n \geq 2r$. Indeed, the generic n -cells of lowest n appearing in $\mathbf{B}^r(\mathcal{Z}M)$ but not in $\mathbf{B}^{r-1}(\mathcal{Z}M)$ are those generic $2r$ -cells of the form $[x_1|_r x_2]$. Thus, via the suspension morphism, $\mathbf{B}^{r-1}(\mathcal{Z}M)_{n-1}$ is identified with $\mathbf{B}^r(\mathcal{Z}M)_n$ for $r \leq n < 2r$, while $\mathbf{B}^{r-1}(\mathcal{Z}M)_{n-1} \subsetneq \mathbf{B}^r(\mathcal{Z}M)_n$ for $n \geq 2r$. In particular, we have the commutative diagram of suspensions

$$\begin{array}{ccccccccc} \mathbf{B}(\mathcal{Z}M)_4 & \longrightarrow & \mathbf{B}(\mathcal{Z}M)_3 & \longrightarrow & \mathbf{B}(\mathcal{Z}M)_2 & \longrightarrow & \mathbf{B}(\mathcal{Z}M)_1 & \longrightarrow & \mathbf{B}(\mathcal{Z}M)_0 \\ \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow \\ \mathbf{B}^2(\mathcal{Z}M)_5 & \longrightarrow & \mathbf{B}^2(\mathcal{Z}M)_4 & \longrightarrow & \mathbf{B}^2(\mathcal{Z}M)_3 & \longrightarrow & \mathbf{B}^2(\mathcal{Z}M)_2 & \longrightarrow & 0 \\ \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \\ \mathbf{B}^3(\mathcal{Z}M)_6 & \longrightarrow & \mathbf{B}^3(\mathcal{Z}M)_5 & \longrightarrow & \mathbf{B}^3(\mathcal{Z}M)_4 & \longrightarrow & \mathbf{B}^3(\mathcal{Z}M)_3 & \longrightarrow & 0 \\ \downarrow S^{r-3} & & \downarrow S^{r-3} & & \downarrow S^{r-3} & & \downarrow S^{r-3} & & \\ \mathbf{B}^r(\mathcal{Z}M)_{r+3} & \longrightarrow & \mathbf{B}^r(\mathcal{Z}M)_{r+2} & \longrightarrow & \mathbf{B}^r(\mathcal{Z}M)_{r+1} & \longrightarrow & \mathbf{B}^r(\mathcal{Z}M)_r & \longrightarrow & 0 \end{array}$$

where in the bottom row is $r \geq 3$, and

- $\mathbf{B}^2(\mathcal{Z}M)_4$ is the free $\mathbb{H}M$ -module on the set of suspensions of the non-zero generic 3-cells $[x_1|x_2|x_3]$ of $\mathbf{B}(\mathcal{Z}M)$ together the non-zero generic 4-cells

$$[x_1\|x_2],$$

with $\pi[x_1\|x_2] = x_1x_2$, and whose differential is $(x = x_1x_2)$

$$\partial_x [x_1\|x_2] = [x_1|x_2] - [x_2|x_1].$$

- $\mathbf{B}^2(\mathcal{Z}M)_5$ is the free $\mathbb{H}M$ -module on the set of suspensions of the non-zero generic 4-cells $[x_1|x_2|x_3|x_4]$ of $\mathbf{B}(\mathcal{Z}M)$ together the non-zero generic 5-cells

$$[x_1\|x_2|x_3], [x_1|x_2\|x_3],$$

with $\pi[x_1\|x_2|x_3] = x_1x_2x_3 = \pi[x_1|x_2\|x_3]$, and whose differential is ($x = x_1x_2x_3$)

$$\begin{aligned}\partial_x[x_1\|x_2|x_3] &= -x_{2*}[x_1\|x_3] + [x_1\|x_2x_3] - x_{3*}[x_1\|x_2] \\ &\quad + [x_1|x_2|x_3] - [x_2|x_1|x_3] + [x_2|x_3|x_1],\end{aligned}$$

$$\begin{aligned}\partial_x[x_1|x_2\|x_3] &= -x_{1*}[x_2\|x_3] + [x_1x_2\|x_3] - x_{2*}[x_1\|x_3] \\ &\quad - [x_1|x_2|x_3] + [x_1|x_3|x_2] - [x_3|x_1|x_2].\end{aligned}$$

• $\mathbf{B}^3(\mathcal{Z}M)_6$ is the free $\mathbb{H}M$ -module on the set of double suspensions of the non-zero generic 4-cells $[x_1|x_2|x_3|x_4]$ of $\mathbf{B}(\mathcal{Z}M)$, together with the suspensions of the non-zero generic 5-cells $[x_1\|x_2|x_3]$ and $[x_1|x_2\|x_3]$ of $\mathbf{B}^2(\mathcal{Z}M)$, and the non-zero generic 6-cells

$$[x_1\|\|x_2],$$

with $\pi[x_1\|\|x_2] = x_1x_2$, whose differential is ($x = x_2x_2$)

$$\partial_x[x_1\|\|x_2] = -[x_1\|x_2] - [x_2\|x_1].$$

Therefore, from Proposition 4.1, we get the following.

Theorem 5.4. For any $\mathbb{H}M$ -module \mathcal{A} , the cohomology groups $H^n(M, r; \mathcal{A})$, for $n \leq r+2$, are isomorphic to the cohomology groups of the truncated cochain complexes of normalized r th level cochains of M with values in \mathcal{A} , $C(M, r; \mathcal{A})$,

$$(19) \quad \begin{array}{ccccccc} C(M, r; \mathcal{A}) : & 0 & \longrightarrow & C^0(M, r; \mathcal{A}) & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow 0 & \longrightarrow & C^r(M, r; \mathcal{A}) \\ & & & & & & & \nwarrow & \\ & & & & & & & C^{r+1}(M, r; \mathcal{A}) & \longrightarrow C^{r+2}(M, r; \mathcal{A}) & \longrightarrow C^{r+3}(M, r; \mathcal{A}) \end{array}$$

where $C^0(M, r; \mathcal{A}) = \mathcal{A}(e)$, and the remaining non-trivial parts occur in the commutative diagram

$$(20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C^1(M, 1; \mathcal{A}) & \longrightarrow & C^2(M, 1; \mathcal{A}) & \longrightarrow & C^3(M, 1; \mathcal{A}) & \longrightarrow & C^4(M, 1; \mathcal{A}) \\ & & \parallel & & \parallel & & \uparrow s^* & & \uparrow s^* \\ 0 & \longrightarrow & C^2(M, 2; \mathcal{A}) & \longrightarrow & C^3(M, 2; \mathcal{A}) & \longrightarrow & C^4(M, 2; \mathcal{A}) & \longrightarrow & C^5(M, 2; \mathcal{A}) \\ & & \parallel & & \parallel & & \parallel & & \uparrow s^* \\ 0 & \longrightarrow & C^3(M, 3; \mathcal{A}) & \longrightarrow & C^4(M, 3; \mathcal{A}) & \longrightarrow & C^5(M, 3; \mathcal{A}) & \longrightarrow & C^6(M, 3; \mathcal{A}) \\ & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & C^r(M, r; \mathcal{A}) & \longrightarrow & C^{r+1}(M, r; \mathcal{A}) & \longrightarrow & C^{r+2}(M, r; \mathcal{A}) & \longrightarrow & C^{r+3}(M, r; \mathcal{A}) \end{array}$$

where in the bottom row is $r \geq 3$, and

• $C^4(M, 2; \mathcal{A})$ is the abelian group, under pointwise addition, of pairs of functions (g, μ) , where

$$g : M^3 \rightarrow \bigcup_{x \in M} \mathcal{A}(x) \quad \mu : M^2 \rightarrow \bigcup_{x \in M} \mathcal{A}(x),$$

with $g(x, y, z) \in \mathcal{A}(xyz)$ and $\mu(x, y) \in \mathcal{A}(xy)$, which are normalized in the sense that they take the value 0 whenever some of their arguments are equal to the unit e of the monoid.

• The coboundary $\partial : C^3(M, 2; \mathcal{A}) = C^2(M, 1; \mathcal{A}) \rightarrow C^4(M, 2; \mathcal{A})$ acts on a normalized 2-cochain f of M in \mathcal{A} by $\partial f = (g, \mu)$, where

$$\begin{aligned}g(x, y, z) &= -x_*f(y, z) + f(xy, z) - f(x, yz) + z_*f(xy), \\ \mu(x, y) &= f(x, y) - f(y, x).\end{aligned}$$

- $C^5(M, 2; \mathcal{A})$ is the abelian group of triples (h, γ, δ) consisting of normalized functions

$$h : M^4 \rightarrow \bigcup_{x \in M} \mathcal{A}(x) \quad \gamma, \delta : M^3 \rightarrow \bigcup_{x \in M} \mathcal{A}(x),$$

with $h(x, y, z, t) \in \mathcal{A}(xyzt)$ and $\gamma(x, y, z), \delta(x, y, z) \in \mathcal{A}(xyz)$.

- The coboundary $\partial : C^4(M, 2; \mathcal{A}) \rightarrow C^5(M, 2; \mathcal{A})$ acts on a 2nd level 4-cochain (g, μ) by $\partial(g, \mu) = (h, \gamma, \delta)$, where

$$\begin{aligned} h(x, y, z, t) &= -x_*g(y, z, t) + g(xy, z, t) - g(x, yz, t) + g(x, y, zt) - t_*g(x, y, z), \\ \gamma(x, y, z) &= -y_*\mu(x, z) + \mu(x, yz) - z_*\mu(x, y) + g(x, y, z) - g(y, x, z) + g(y, z, x), \\ \delta(x, y, z) &= -x_*\mu(y, z) + \mu(xy, z) - y_*\mu(x, z) - g(x, y, z) + g(x, z, y) - g(z, x, y). \end{aligned}$$

- $C^6(M, 3; \mathcal{A})$ is the abelian group of quadruples (h, γ, δ, ξ) consisting of normalized functions

$$h : M^4 \rightarrow \bigcup_{x \in M} \mathcal{A}(x), \quad \gamma, \delta : M^3 \rightarrow \bigcup_{x \in M} \mathcal{A}(x), \quad \xi : M^2 \rightarrow \bigcup_{x \in M} \mathcal{A}(x),$$

with $h(x, y, z, t) \in \mathcal{A}(xyzt)$, $\gamma(x, y, z), \delta(x, y, z) \in \mathcal{A}(xyz)$, and $\xi(x, y) \in \mathcal{A}(xy)$.

- The coboundary $\partial : C^5(M, 3; \mathcal{A}) = C^4(M, 2; \mathcal{A}) \rightarrow C^6(M, 3; \mathcal{A})$ acts on a 3rd-level 5-cochain by $\partial(g, \mu) = (h, \gamma, \delta, \xi)$, where

$$\begin{aligned} h(x, y, z, t) &= x_*g(y, z, t) - g(xy, z, t) + g(x, yz, t) - g(x, y, zt) + t_*g(x, y, z), \\ \gamma(x, y, z) &= y_*\mu(x, z) - \mu(x, yz) + z_*\mu(x, y) - g(x, y, z) + g(y, x, z) - g(y, z, x), \\ \delta(x, y, z) &= x_*\mu(y, z) - \mu(xy, z) + y_*\mu(x, z) + g(x, y, z) - g(x, z, y) + g(z, x, y) \\ \xi(x, y) &= -\mu(x, y) - \mu(y, x). \end{aligned}$$

The following corollaries follow directly from the form of the cochain complex (19) and the commutativity of the diagram (20).

Corollary 5.5. For any $r \geq 1$, $H^0(M, r; \mathcal{A}) \cong \mathcal{A}(e)$.

Corollary 5.6. For any $0 < n < r$, $H^n(M, r; \mathcal{A}) = 0$.

Corollary 5.7. For any $r \geq 2$, $H^r(M, r; \mathcal{A}) \cong H^1(M, 1; \mathcal{A})$.

Corollary 5.8. For any $r \geq 2$, $H^{r+1}(M, r; \mathcal{A}) \cong H^3(M, 2; \mathcal{A})$, and there is a natural monomorphism $H^3(M, 2; \mathcal{A}) \hookrightarrow H^2(M, 1; \mathcal{A})$.

Corollary 5.9. For any $r \geq 3$, $H^{r+2}(M, r; \mathcal{A}) \cong H^5(M, 3; \mathcal{A})$, and there is a natural monomorphism $H^5(M, 3; \mathcal{A}) \hookrightarrow H^4(M, 2; \mathcal{A})$.

Let us now recall that the so-called Grillet cohomology groups $H_G^n(M, \mathcal{A})$, for $1 \leq n \leq 3$, can be computed as the cohomology groups of the truncated cochain complex $C_G(M, \mathcal{A})$,

$$0 \rightarrow C_G^1(M, \mathcal{A}) \xrightarrow{\delta^1} C_G^2(M, \mathcal{A}) \xrightarrow{\delta^2} C_G^3(M, \mathcal{A}) \xrightarrow{\delta^3} C_G^4(M, \mathcal{A}),$$

called the complex of (normalized on $e \in M$) *symmetric cochains* on M with values in \mathcal{A} [12, Chapters XII, XIII, XIV], where

- $C_G^1(M, \mathcal{A})$ consists of normalized functions $f : M \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$, with $f(x) \in \mathcal{A}(x)$.
- $C_G^2(M, \mathcal{A})$ consists of normalized functions $f : M^2 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$, with $f(x, y) \in \mathcal{A}(xy)$, such that $f(x, y) = f(y, x)$.
- $C_G^3(M, \mathcal{A})$ consists of normalized functions $f : M^3 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$ with $f(x, y, z) \in \mathcal{A}(xyz)$, such that

$$f(x, y, z) + f(z, y, x) = 0, \quad f(x, y, z) + f(y, z, x) + f(z, x, y) = 0.$$

• $C_G^4(M, \mathcal{A})$ consists of normalized functions $f : M^4 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$ with $f(x, y, z, t) \in \mathcal{A}(xyzt)$, such that

$$\begin{aligned} f(x, y, y, x) &= 0, & f(t, z, y, x) + f(x, y, z, t) &= 0, \\ f(x, y, z, t) - f(y, z, t, x) + f(z, t, x, y) - f(t, x, y, z) &= 0, \\ f(x, y, z, t) - f(y, x, z, t) + f(y, z, x, t) - f(y, z, t, x) &= 0. \end{aligned}$$

• the coboundary homomorphisms are defined by

$$\begin{cases} (\delta^1 f)(x, y) = -x_* f(y) + f(xy) - y_* f(x), \\ (\delta^2 f)(x, y, z) = -x_* f(y, z) + f(xy, z) - f(x, yz) + z_* f(x, y), \\ (\delta^3 f)(x, y, z, t) = -x_* f(y, z, t) + f(xy, z, t) - f(x, yz, t) + f(x, y, zt) - t_* f(x, y, z). \end{cases}$$

There is natural injective cochain map

$$(21) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_G^1(M, \mathcal{A}) & \xrightarrow{\delta^1} & C_G^2(M, \mathcal{A}) & \xrightarrow{\delta^2} & C_G^3(M, \mathcal{A}) & \xrightarrow{\delta^3} & C_G^4(M, \mathcal{A}) \\ & & \parallel i_1 = id & & \downarrow i_2 & & \downarrow i_3 & & \downarrow i_4 \\ 0 & \longrightarrow & C^3(M, 3; \mathcal{A}) & \xrightarrow{\partial^3} & C^4(M, 3; \mathcal{A}) & \xrightarrow{\partial^4} & C^5(M, 3; \mathcal{A}) & \xrightarrow{\partial^5} & C^6(M, 3; \mathcal{A}), \end{array}$$

which is the identity map, $i_1(f) = f$, on symmetric 1-cochains, the map $i_2(f) = -f$ on symmetric 2-cochains, and on symmetric 3- and 4-cochains is defined by the simple formulas $i_3(f) = (f, 0)$ and $i_4(f) = (-f, 0, 0, 0)$, respectively. The only non-trivial verification here concerns the equality $\partial^5 i_3 = i_4 \delta^3$, that is, $\partial^5(f, 0) = (-\delta^3 f, 0, 0, 0)$, for any $f \in C_G^3(M, \mathcal{A})$, but it easily follows from Lemma 5.10 below.

Lemma 5.10. *Let $f : M^3 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$ be a function with $f(x, y, z) \in \mathcal{A}(xyz)$. Then f satisfies the symmetry conditions*

$$(22) \quad f(x, y, z) + f(z, y, x) = 0, \quad f(x, y, z) + f(y, z, x) + f(z, x, y) = 0,$$

if and only if it satisfies either (23) or (24) below.

$$(23) \quad f(x, y, z) - f(y, x, z) + f(y, z, x) = 0$$

$$(24) \quad f(x, y, z) - f(x, z, y) + f(z, x, y) = 0$$

Proof. The implications $(22) \Rightarrow (23)$ and $(22) \Rightarrow (24)$ are easily seen. To see that $(23) \Rightarrow (22)$, observe that, making the permutation $(x, y, z) \mapsto (z, y, x)$, equation (23) is written as $f(y, z, x) = f(z, y, x) + f(y, x, z)$. If we carry this to (23), we obtain

$$f(x, y, z) - f(y, x, z) + f(z, y, x) + f(y, x, z) = f(x, y, z) + f(z, y, x) = 0,$$

that is, the first condition in (22) holds. But then, we get also the second one simply by replacing the term $f(y, x, z)$ with $-f(z, x, y)$ in (23). The proof that $(24) \Rightarrow (22)$ is parallel. \square

Proposition 5.11. *For any $\mathbb{H}M$ -module \mathcal{A} , the injective cochain map (21) induces natural isomorphisms*

$$H_G^1(M, \mathcal{A}) \cong H^1(M, 1; \mathcal{A}), \quad H_G^2(M, \mathcal{A}) \cong H^3(M, 2; \mathcal{A}),$$

and a natural monomorphism

$$H_G^3(M, \mathcal{A}) \hookrightarrow H^5(M, 3; \mathcal{A}).$$

Proof. From diagram (21), it follows directly that $\ker \delta^1 = \ker \partial^3$ and $i_2 \text{Im } \delta^1 = \text{Im } \partial^3$. Further, $i_2 \ker \delta^2 = \ker \partial^4$, since the condition $\partial^4 f = 0$ on a cochain $f \in C^4(M, 3; \mathcal{A}) = C^2(M, 1; \mathcal{A})$ implies the symmetry condition $f(x, y) = f(y, x)$. Then,

$$H_G^1(M, \mathcal{A}) = \ker \delta^1 = \ker \partial^3 \cong H^3(M, 3; \mathcal{A}) \cong H^1(M, 1; \mathcal{A}),$$

and

$$H_G^2(M, \mathcal{A}) = \frac{\ker \delta^2}{\operatorname{Im} \delta^1} \cong \frac{i_2 \ker \delta^2}{i_2 \operatorname{Im} \delta^1} = \frac{\ker \partial^4}{\operatorname{Im} \partial^3} \cong H^4(M, 3; \mathcal{A}) \cong H^3(M, 2; \mathcal{A}).$$

To prove that the induced homomorphism $H_G^3(M, \mathcal{A}) \rightarrow H^5(M, 3; \mathcal{A})$ is injective, suppose $f \in C_G^3(M, \mathcal{A})$ is a symmetric 3-cochain such that $i_3 f = \partial^4 g$ for some $g \in C^4(M, 3; \mathcal{A}) = C^2(M, 1; \mathcal{A})$. This means that the equalities

$$f(x, y, z) = x_* g(y, z) - g(xy, z) + g(x, yz) - z_* g(x, y), \quad 0 = g(x, y) - g(y, x),$$

hold. Then, $g \in C_G^2(M, \mathcal{A})$ is a symmetric 2-cochain, and $f = -\delta^2 g$ is actually a symmetric 2-coboundary. It follows that the injective map $i_3 : \ker \delta^3 \hookrightarrow \ker \partial^5$ induces an injective map in cohomology $H^3 C_G(M, \mathcal{A}) \hookrightarrow H^5 C(M, 3; \mathcal{A})$, as required. \square

To complete the list of relationships between the cohomology groups $H^n(M, r; \mathcal{A})$ with those already known in the literature, let us note that a direct comparison of the cochain complex (19) with the cochain complex in [2, (6)], which computes the lower *commutative cohomology groups* $H_C^n(M, \mathcal{A})$, gives the following.

Proposition 5.12. *For any $\mathbb{H}M$ -module \mathcal{A} , there are natural isomorphisms*

$$H^1(M, 1; \mathcal{A}) \cong H_C^1(M, \mathcal{A}), \quad H^3(M, 2; \mathcal{A}) \cong H_C^2(M, \mathcal{A}), \quad H^4(M, 2; \mathcal{A}) \cong H_C^3(M, \mathcal{A}).$$

6. COHOMOLOGY CLASSIFICATION OF SYMMETRIC MONOIDAL ABELIAN GROUPOIDS

This section is dedicated to showing a precise classification for symmetric monoidal abelian groupoids, by means of the 3rd level cohomology groups of commutative monoids $H^5(M, 3; \mathcal{A})$.

Symmetric monoidal categories have been studied extensively in the literature and we refer to Mac Lane [17] and Saavedra [20] for the background. Recall that a *groupoid* is a small category all whose morphisms are invertible. A groupoid \mathcal{M} is said to be *abelian* if its isotropy (or vertex) groups $\operatorname{Aut}_{\mathcal{M}}(x)$, $x \in \operatorname{Ob} \mathcal{M}$, are all abelian. We will use additive notation for abelian groupoids. Thus, the identity morphism of an object x of an abelian groupoid \mathcal{M} will be denoted by 0_x ; if $a : x \rightarrow y$, $b : y \rightarrow z$ are morphisms, their composite is written by $b + a : x \rightarrow z$, while the inverse of a is $-a : y \rightarrow x$.

A *symmetric monoidal abelian groupoid*

$$\mathcal{M} = (\mathcal{M}, \otimes, \mathbf{I}, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{c})$$

consists of an abelian groupoid \mathcal{M} , a functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ (the *tensor product*), an object \mathbf{I} (the *unit object*), and natural isomorphisms $\mathbf{a}_{x,y,z} : (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$, $\mathbf{l}_x : \mathbf{I} \otimes x \rightarrow x$, $\mathbf{r}_x : x \otimes \mathbf{I} \rightarrow x$ (called the *associativity* and *unit constraints*, respectively) and $\mathbf{c}_{x,y} : x \otimes y \rightarrow y \otimes x$ (the *symmetry*), such that the four coherence conditions below hold.

$$(25) \quad \mathbf{a}_{x,y,z \otimes t} + \mathbf{a}_{x \otimes y, z, t} = (0_x \otimes \mathbf{a}_{y,z,t}) + \mathbf{a}_{x, y \otimes z, t} + (\mathbf{a}_{x,y,z} \otimes 0_t),$$

$$(26) \quad (0_x \otimes \mathbf{l}_y) + \mathbf{a}_{x, \mathbf{I}, y} = \mathbf{r}_x \otimes 0_y,$$

$$(27) \quad (0_y \otimes \mathbf{c}_{x,z}) + \mathbf{a}_{y, x, z} + (\mathbf{c}_{x,y} \otimes 0_z) = \mathbf{a}_{y, z, x} + \mathbf{c}_{x, y \otimes z} + \mathbf{a}_{x, y, z},$$

$$(28) \quad \mathbf{c}_{y,x} + \mathbf{c}_{x,y} = 0_{x \otimes y}.$$

For further use, we recall that, in any symmetric monoidal abelian groupoid \mathcal{M} , the equalities below hold (see [14, Propositions 1.1 and 2.1]).

$$(29) \quad \mathbf{l}_{x \otimes y} + \mathbf{a}_{\mathbf{I}, x, y} = \mathbf{l}_x \otimes 0_y, \quad 0_x \otimes \mathbf{r}_y + \mathbf{a}_{x, y, \mathbf{I}} = \mathbf{r}_{x \otimes y},$$

$$(30) \quad \mathbf{l}_x + \mathbf{c}_{x, \mathbf{I}} = \mathbf{r}_x, \quad \mathbf{r}_x + \mathbf{c}_{\mathbf{I}, x} = \mathbf{l}_x.$$

Example 6.1 (*2-dimensional crossed products*). Every 3rd level 5-cocycle $(g, \mu) \in Z^5(M, 3; \mathcal{A})$, gives rise to a symmetric monoidal abelian groupoid

$$\mathcal{A} \rtimes_{g, \mu} M = (\mathcal{A} \rtimes_{g, \mu} M, \otimes, \mathbf{I}, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{c}),$$

that should be thought of as a *2-dimensional crossed product of M by \mathcal{A}* , and it is built as follows: Its underlying groupoid has as set of objects the set M ; if $x \neq y$ are different elements of the monoid M , then there are no morphisms in $\mathcal{A} \rtimes_{g, \mu} M$ between them, whereas its isotropy group at any $x \in M$ is $\mathcal{A}(x)$.

The tensor product $\otimes : (\mathcal{A} \rtimes_{g, \mu} M) \times (\mathcal{A} \rtimes_{g, \mu} M) \rightarrow \mathcal{A} \rtimes_{g, \mu} M$ is given on objects by multiplication in M , so $x \otimes y = xy$, and on morphisms by the group homomorphisms

$$\otimes : \mathcal{A}(x) \times \mathcal{A}(y) \rightarrow \mathcal{A}(xy), \quad a_x \otimes a_y = y_* a_x + y_* a_y.$$

The unit object is $\mathbf{I} = e$, the unit of the monoid M , and the structure constraints are

$$\begin{aligned} \mathbf{a}_{x,y,z} &= g(x, y, z) : (xy)z \rightarrow x(yz), \\ \mathbf{c}_{x,y} &= \mu(x, y) : xy \rightarrow yx, \\ \mathbf{l}_x &= 0_x : ex = x \rightarrow x \\ \mathbf{r}_x &= 0_x : xe = x \rightarrow x, \end{aligned}$$

which are easily seen to be natural since \mathcal{A} is an abelian group valued functor. The coherence conditions (25), (27), and (28) follow from the 5-cocycle condition $\partial^5(h, \mu) = (0, 0, 0, 0)$, while the coherence condition (26) holds due to the normalization conditions $h(x, e, y) = 0$.

If $\mathcal{M}, \mathcal{M}'$ are symmetric monoidal abelian groupoids, then a *symmetric monoidal functor* $F = (F, \varphi, \varphi_0) : \mathcal{M} \rightarrow \mathcal{M}'$ consists of a functor between the underlying groupoids $F : \mathcal{M} \rightarrow \mathcal{M}'$, natural isomorphisms $\varphi_{x,y} : Fx \otimes' Fy \rightarrow F(x \otimes y)$, and an isomorphism $\varphi_0 : \mathbf{I}' \rightarrow F\mathbf{I}$, such that the following coherence conditions hold:

$$(31) \quad \varphi_{x,y \otimes z} + (0_{Fx} \otimes' \varphi_{y,z}) + \mathbf{a}'_{Fx, Fy, Fz} = F(\mathbf{a}_{x,y,z}) + \varphi_{x \otimes y, z} + (\varphi_{x,y} \otimes' 0_{Fz}),$$

$$(32) \quad F(\mathbf{l}_x) + \varphi_{\mathbf{I}, x} + (\varphi_0 \otimes' 0_{Fx}) = \mathbf{l}'_{Fx}, \quad F(\mathbf{r}_x) + \varphi_{x, \mathbf{I}} + (0_{Fx} \otimes' \varphi_0) = \mathbf{r}'_{Fx},$$

$$(33) \quad \varphi_{y,x} + \mathbf{c}'_{Fx, Fy} = F(\mathbf{c}_{x,y}) + \varphi_{x,y}.$$

Suppose $F' : \mathcal{M} \rightarrow \mathcal{M}'$ is another symmetric monoidal functor. Then, a *symmetric monoidal isomorphism* $\theta : F \Rightarrow F'$ is a natural isomorphism between the underlying functors, $\theta : F \Rightarrow F'$, such that the following coherence conditions hold:

$$\varphi'_{x,y} + (\theta_x \otimes' \theta_y) = \theta_{x \otimes y} + \varphi_{x,y}, \quad \theta_{\mathbf{I}} + \varphi_0 = \varphi'_0.$$

With compositions defined in a natural way, symmetric monoidal abelian groupoids, symmetric monoidal functors, and symmetric monoidal isomorphisms form a 2-category [10, Chaper V, §1]. A symmetric monoidal functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ is called a *symmetric monoidal equivalence* if it is an equivalence in this 2-category.

Our goal is to show a classification for symmetric monoidal abelian groupoids, where two symmetric monoidal abelian groupoids connected by a symmetric monoidal equivalence are considered the same, as stated in the theorem below. Recall that any homomorphism of monoids $i : M \rightarrow M'$ induces a functor $\mathbb{H}M \rightarrow \mathbb{H}M'$ in a obvious way, and then, by composition with it, a functor $i^* : \mathbb{H}M' \text{-Mod} \rightarrow \mathbb{H}M \text{-Mod}$.

Theorem 6.2 (Classification of Symmetric Monoidal Abelian Groupoids). *(i) For any symmetric monoidal abelian groupoid \mathcal{M} , there exist a commutative monoid M , a $\mathbb{H}M$ -module \mathcal{A} , a 3rd level 5-cocycle $(g, \mu) \in Z^5(M, 3; \mathcal{A})$, and a symmetric monoidal equivalence*

$$\mathcal{A} \rtimes_{g, \mu} M \simeq \mathcal{M}.$$

(ii) For any two 3rd level 5-cocycles $(g, \mu) \in Z^5(M, 3; \mathcal{A})$ and $(g', \mu') \in Z^5(M', 3; \mathcal{A}')$, there is a symmetric monoidal equivalence

$$\mathcal{A} \rtimes_{g, \mu} M \simeq \mathcal{A}' \rtimes_{g', \mu'} M'$$

if and only if there exist an isomorphism of monoids $i : M \cong M'$ and a natural isomorphism $\psi : \mathcal{A} \cong i^* \mathcal{A}'$, such that the equality of cohomology classes below holds.

$$[g, \mu] = \psi_*^{-1} i^* [g', \mu'] \in H^5(M, 3; \mathcal{A})$$

Proof. (i) Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathbf{I}, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{c})$ be any given symmetric monoidal abelian groupoid.

By the coherence theorem [17], there is no loss of generality in assuming that \mathcal{M} is itself strictly unitary, that is, where both unit constraints \mathbf{l} and \mathbf{r} are identities. Then, we observe that \mathcal{M} is symmetric monoidal equivalent to another one that is totally disconnected, that is, where there is no morphism between different objects. Indeed, by the generalized Brandt's theorem [13, Chapter 6, Theorem 2], there is a totally disconnected groupoid, say \mathcal{M}' , with an equivalence of groupoids $\mathcal{M} \rightarrow \mathcal{M}'$. Hence, by Saavedra [20, I, 4.4], we can transport the symmetric monoidal structure along this equivalence so that \mathcal{M}' becomes a strictly unitary symmetric monoidal abelian groupoid and the equivalence a symmetric monoidal one.

Hence, we assume that \mathcal{M} is totally disconnected and strictly unitary. Then, a triplet $(M, \mathcal{A}, (g, \mu))$, such that $\mathcal{A} \rtimes_{g, \mu} M = \mathcal{M}$ as symmetric monoidal abelian groupoids, can be defined as follows:

- *The monoid M .* Let $M = \text{Ob}\mathcal{M}$ be the set of objects of \mathcal{M} . The tensor functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ determines a multiplication on M , simply by $xy = x \otimes y$, for any $x, y \in M$. Since \mathcal{M} is strictly unitary, this multiplication on M is unitary with $e = \mathbf{I}$, the unit object of \mathcal{M} . Moreover, it is associative and commutative since \mathcal{M} being totally disconnected implies that $(xy)z = x(yz)$ and $xy = yx$. Thus, M becomes a commutative monoid.

- *The $\mathbb{H}M$ -module \mathcal{A} .* For each $x \in M = \text{Ob}\mathcal{M}$, let $\mathcal{A}(x) = \text{Aut}_{\mathcal{M}}(x)$ be the vertex group of the underlying abelian groupoid at x . Since the diagrams below commute, due to the naturality of the structure constraints and the symmetry,

$$\begin{array}{ccccc} (xy)z & \xrightarrow{\mathbf{a}_{x,y,z}} & x(yz) & & xy & \xrightarrow{\mathbf{c}_{x,y}} & yx & & ex = x & \xrightarrow{0_x} & x \\ (a_x \otimes a_y) \otimes a_z \downarrow & & \downarrow a_x \otimes (a_y \otimes a_z) & & a_x \otimes a_y \downarrow & & \downarrow a_y \otimes a_x & & 0_e \otimes a_x \downarrow & & \downarrow a_x \\ (xy)z & \xrightarrow{\mathbf{a}_{x,y,z}} & x(yz) & & xy & \xrightarrow{\mathbf{c}_{x,y}} & yx & & ex = x & \xrightarrow{0_x} & x \end{array}$$

it follows that the equations below hold.

$$(34) \quad (a_x \otimes a_y) \otimes a_z = a_x \otimes (a_y \otimes a_z), \quad a_x \otimes a_y = a_y \otimes a_x, \quad 0_e \otimes a_x = a_x,$$

Then, if we write $y_* : \mathcal{A}(x) \rightarrow \mathcal{A}(xy)$ for the homomorphism such that

$$y_* a_x = 0_y \otimes a_x = a_x \otimes 0_y,$$

the equalities

$$\begin{aligned} (yz)_*(a_x) &= 0_{yz} \otimes a_x = (0_y \otimes 0_z) \otimes a_x \stackrel{(34)}{=} 0_y \otimes (0_z \otimes a_x) = y_*(z_* a_x), \\ e_* a_x &= 0_e \otimes a_x \stackrel{(34)}{=} a_x, \end{aligned}$$

implies that the assignments $x \mapsto \mathcal{A}(x)$, $(x, y) \mapsto y_* : \mathcal{A}(x) \rightarrow \mathcal{A}(xy)$, define an $\mathbb{H}M$ -module. Observe that this $\mathbb{H}M$ -module \mathcal{A} determines indeed the tensor product \otimes of \mathcal{M} , since

$$a_x \otimes a_y = (a_x + 0_x) \otimes (0_y + a_y) = (a_x \otimes 0_y) + (0_x \otimes a_y) = y_* a_x + x_* a_y.$$

- *The 3rd level 5-cocycle $(g, \mu) \in Z^5(M, 3; \mathcal{A})$.* The associativity constraint and the symmetry of \mathcal{M} can be written in the form $\mathbf{a}_{x,y,z} = g(x, y, z)$ and $\mathbf{c}_{x,y} = \mu(x, y)$, for some given lists $(g(x, y, z) \in \mathcal{A}(xyz))_{x,y,z \in M}$ and $(\mu(x, y) \in \mathcal{A}(xy))_{x,y \in M}$. Since \mathcal{M} is strictly unitary, equations

in (26) and (29) give the normalization conditions $g(x, e, y) = 0 = g(e, x, y) = g(x, y, e)$ for g , while equations in (30) imply the normalization conditions $\mu(x, e) = 0 = \mu(e, x)$ for μ . Thus, $(g, \mu) \in C^5(M, 3; \mathcal{A})$ is a 3rd level 5-cochain. By the coherence conditions (25), (27), and (28) we have that

$$\begin{aligned} g(x, y, zt) + g(xy, z, y) &= x_*g(y, z, y) + g(x, yz, y) + t_*g(x, y, z) \\ y_*\mu(x, z) + g(y, x, z) + z_*\mu(x, y) &= g(y, z, x) + \mu(x, yz) + g(x, y, z), \\ \mu(x, y) + \mu(y, x) &= 0, \end{aligned}$$

and combining the last two equations we also have

$$-y_*\mu(z, x) + g(y, x, z) - z_*\mu(y, x) = g(y, z, x) - \mu(yz, x) + g(x, y, z).$$

Hence, we obtain the required cocycle condition $\partial^3(g, \mu) = (0, 0, 0)$. Since a direct comparison shows that $\mathcal{M} = \mathcal{A} \rtimes_{g, \mu} M$ as symmetric monoidal abelian groupoids, the proof of this part is complete.

(ii) Suppose there exist an isomorphism of monoids $i : M \cong M'$ and a natural isomorphism $\psi : \mathcal{A} \cong i^* \mathcal{A}'$ such that $\psi_*[g, \mu] = i^*[g', \mu'] \in H^5(M, 3; i^* \mathcal{A}')$. This implies that there is a 3rd level 4-cochain $f \in C^4(M, 3; i^* \mathcal{A}') = C^2(M, 1; i^* \mathcal{A}')$ such that

$$(35) \quad \psi_{xyz}g(x, y, z) = g'(ix, iy, iz) + (ix)_*f(y, z) - f(xy, z) + f(x, yz) - (iz)_*f(x, y),$$

$$(36) \quad \psi_{xy}\mu(x, y) = \mu'(ix, iy) - f(x, y) + f(y, x).$$

Then, a symmetric monoidal isomorphism

$$F(f) = (F, \varphi, \varphi_0) : \mathcal{A} \rtimes_{g, \mu} M \rightarrow \mathcal{A}' \rtimes_{g', \mu'} M'$$

can be defined as follows: The underlying functor acts by $F(a_x : x \rightarrow x) = (\psi_x a_x : ix \rightarrow ix)$. The constraints of F are given by $\varphi_{x, y} = f(x, y) : (ix)(iy) \rightarrow i(xy)$, and $\varphi_0 = 0_e : e \rightarrow ie = e$. So defined, it is easy to see that F is an isomorphism between the underlying groupoids. The naturality of the isomorphisms $\varphi_{x, y}$, that is,

$$(37) \quad \psi_{xy}(x_*a_y + y_*a_x) + \varphi_{x, y} = \varphi_{x, y} + (ix)_*\psi_y a_y + (iy)_*\psi_x a_x$$

for $a_x \in \mathcal{A}(x)$, $a_y \in \mathcal{A}(y)$, holds owing to the commutativity of $\mathcal{A}'(i(xy))$ and the naturality of $\psi : \mathcal{A} \cong i^* \mathcal{A}'$, which says that

$$(38) \quad \psi_{xy}(x_*a_y) = (ix)_*\psi_y a_y.$$

The coherence conditions (31) and (33) are obtained as a consequence of equations (35) and (36), respectively, whereas the conditions in (32) trivially follow from the normalization conditions $f(x, e) = 0_{ix} = f(e, x)$.

Conversely, suppose we have $F = (F, \varphi, \varphi_0) : \mathcal{A} \rtimes_{g, \mu} M \rightarrow \mathcal{A}' \rtimes_{g', \mu'} M'$ a symmetric monoidal equivalence. By [6, Lemma 18], there is no loss of generality in assuming that F is strictly unitary in the sense that $\varphi_0 = 0_e : e \rightarrow e = Fe$. As the underlying functor establishes an equivalence between the underlying groupoids, and these are totally disconnected, it is an isomorphism.

We write $i : M \cong M'$ for the bijection established by F between the object sets; that is, such that $ix = Fx$, $x \in M$. Then, i is actually an isomorphism of monoids, since the existence of the structure isomorphisms $\varphi_{x, y} : (ix)(iy) \rightarrow i(xy)$ implies $(ix)(iy) = i(xy)$.

Let us write now $\psi_x : \mathcal{A}(x) \cong \mathcal{A}'(ix)$ for the isomorphism such that $Fa_x = \psi_x a_x$, for each automorphism $a_x \in \mathcal{A}(x)$, and $x \in M$. The naturality of the automorphisms $\varphi_{x, y}$ tell us that the equalities (37) hold. In particular, when $a_x = 0_x$, we obtain the equation (38) and so $\psi : \mathcal{A} \cong i^* \mathcal{A}'$ is indeed a natural isomorphism.

Finally, if we write $f(x, y) = \varphi_{x, y}$, for each $x, y \in M$, we have a 3rd level 4-cochain $f(F) = (f(x, y) \in \mathcal{A}'(i(xy)))_{x, y \in M}$, since the equations $f(x, e) = 0_{ix} = f(e, x)$ hold due to (32). Equations (35) and (36) follow from the coherence equations (31) and (33). This means

that $\psi_*(g, \mu) = i^*(g', \mu') + \partial^4 f$ and, therefore, we have that $\psi_*[g, \mu] = i^*[g', \mu'] \in H^5(M, 3; i^* \mathcal{A}')$, whence $[g, \mu] = \psi_*^{-1} i^*[g', \mu'] \in H^5(M, 3; \mathcal{A})$. \square

7. COHOMOLOGY OF CYCLIC MONOIDS

In this section we compute the cohomology groups $H^n(C, r; \mathcal{A})$, for $n \leq r + 2$, when C is any cyclic monoid. The method we employ follows similar lines to the one used by Eilenberg and Mac Lane in [8, §14 and §15], for computing higher level cohomology of cyclic groups, though the generalization to monoids is highly nontrivial.

7.1. Cohomology of finite cyclic monoids.

The structure of finite cyclic monoids was first stated by Frobenius [9]. Briefly, let us recall that if \equiv is any not equality congruence on the additive monoid $\mathbb{N} = \{0, 1, \dots\}$ of natural numbers, then the least $m \geq 0$ such that $m \equiv x$ for some $x \neq m$ is called the *index* of the congruence, and the least $q \geq 1$ such that $m \equiv m + q$ is called its *period*. Hence,

$$x \equiv y \text{ if and only if either } x = y < m, \text{ or } x, y \geq m \text{ and } x \equiv y \pmod{q}.$$

The quotient \mathbb{N}/\equiv is called the *cyclic monoid of index m and period q* , and denoted here $C_{m,q}$. As \mathbb{N} is a free monoid on the generator 1, every finite cyclic monoid is isomorphic to a proper quotient of \mathbb{N} and, therefore, to a monoid $C_{m,q}$ for some m and q .

From now on, $C = C_{m,q}$ denotes the finite cyclic monoid of index m and period q . We assume that $m + q \geq 2$, so that C is not the zero monoid.

Since every element of C can be written uniquely in the form $[x]$ with $0 \leq x < m + q$, this monoid can be described as the set

$$C = \{0, 1, \dots, m, m + 1, \dots, m + q - 1\},$$

with addition

$$x \oplus y = \wp(x + y),$$

where $\wp : \mathbb{N} \rightarrow C$ is the projection map given by

$$\wp(x) = \begin{cases} x & \text{if } x < m + q \\ x - kq & \text{if } m + kq \leq x < m + (k + 1)q. \end{cases}$$

To any pair $x, y \in C$, we can associate the useful integer

$$s(x, y) = \frac{(x + y) - (x \oplus y)}{q},$$

which satisfies $s(x, y) \geq 1$ if $x + y \geq m + q$, whereas $s(x, y) = 0$ if $x + y < m + q$. It follows directly from the associativity in C that the cocycle property below holds.

$$(39) \quad s(y, z) + s(x, y \oplus z) = s(x \oplus y, z) + s(x, y).$$

Next, we construct a specific commutative DGA-algebra over $\mathbb{H}C$, denoted by

$$\mathcal{R} = \mathcal{R}(C),$$

which is homologically equivalent to $\mathbf{B}(\mathcal{Z}C)$ but algebraically simpler and more lucid. For each integer $k = 0, 1, \dots$, let us choose unitary sets over C , $\{v_k\}$ and $\{w_k\}$, with

$$(40) \quad \pi v_k = \wp(km), \quad \pi w_k = \wp(km + 1),$$

and define

$$(41) \quad \begin{cases} \mathcal{R}_{2k} & = \text{the free } \mathbb{H}C\text{-module on } \{v_k\}, \\ \mathcal{R}_{2k+1} & = \text{the free } \mathbb{H}C\text{-module on } \{w_k\}. \end{cases}$$

The augmentation $\alpha : \mathcal{R}_0 \rightarrow \mathbb{Z}$, the differential $\partial : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1}$, and the multiplication $\circ : \mathcal{R} \otimes_{\mathbb{H}C} \mathcal{R} \rightarrow \mathcal{R}$ are determined by the equations

$$(42) \quad \alpha v_0 = 1, \quad \partial v_{k+1} = (m+q)((m+q-1)_* w_k) - m((m-1)_* w_k), \quad \partial w_k = 0,$$

$$(43) \quad v_k \circ v_l = \binom{k+l}{k} v_{k+l}, \quad w_k \circ w_l = 0, \quad v_k \circ w_l = \binom{k+l}{k} w_{k+l} = w_l \circ v_k,$$

and the unit is v_0 .

Proposition 7.1. *\mathcal{R} , defined as above, is a commutative DGA-algebra over $\mathbb{H}C$.*

Proof. By Proposition 4.1, the mapping in (42), $v_{k+1} \mapsto \partial v_{k+1}$, determines a morphism of $\mathbb{H}C$ -modules $\partial : \mathcal{R}_{2k+2} \rightarrow \mathcal{R}_{2k+1}$ since

$$\begin{aligned} (m+q-1) \oplus \pi w_k &\stackrel{(40)}{=} (m+q-1) \oplus \wp(km+1) = \wp(m+q+km) = \wp(m+km) = \pi v_{k+1}, \\ (m-1) \oplus \pi w_k &\stackrel{(40)}{=} (m-1) \oplus \wp(km+1) = \wp(m+km) = \pi v_{k+1}, \end{aligned}$$

and therefore $\partial v_{k+1} \in \mathcal{R}_{2k+1}(\pi v_{k+1})$. Similarly, by Proposition 4.1, we see that the formulas in (43) determine a multiplication morphism of $\mathbb{H}C$ -modules since $\wp(km) \oplus \wp(lm) = \wp((k+l)m)$ and $\wp(km) \oplus \wp(lm+1) = \wp((k+l)m+1)$. Associativity condition (8) follows from the equality on combinatorial numbers

$$\binom{k+l+t}{k} + \binom{l+t}{t} = \frac{(k+l+t)!}{k!l!t!} = \binom{k+l+t}{k+l} + \binom{k+l}{l},$$

while condition (9) holds thanks to the equality

$$\binom{k+l-1}{k-1} + \binom{k+l-1}{k} = \binom{k+l}{k},$$

and the remaining conditions in (5)-(7) are quite obviously verified. \square

In next proposition we shall define a morphism $f : \mathbf{B}(\mathcal{Z}C) \rightarrow \mathcal{R}$. Previously, observe that the graded $\mathbb{H}C$ -module $\{\mathcal{R}_n\}$ admits another structure of commutative graded algebra over $\mathbb{H}C$ (although it does not respect the differential structure), whose multiplication is determined by the simpler formulas

$$v_k \bullet v_l = v_{k+l}, \quad w_k \bullet w_l = 0, \quad v_k \bullet w_l = w_{k+l} = w_l \bullet v_k.$$

Proposition 7.2. *A morphism $f : \mathbf{B}(\mathcal{Z}C) \rightarrow \mathcal{R}$, of DGA-algebras over $\mathbb{H}C$, may be defined by the recursive formulas*

$$(44) \quad \left\{ \begin{array}{ll} f[\] &= v_0, \\ f[x] &= x((x-1)_* w_0), \\ f[x \mid y] &= \begin{cases} 0 & \text{if } x+y < m+q, \\ ((x \oplus y)-m)_* \left(\sum_{i=0}^{s(x,y)-1} (iq)_* v_1 \right) & \text{if } x+y \geq m+q, \end{cases} \\ f[x \mid y \mid \sigma] &= f[x \mid y] \bullet f[\sigma], \end{array} \right.$$

where $\sigma = [z \mid \dots]$ is any cell of dimension 1 or greater.

Proof. This is divided into four parts. Note first that, from the inequalities

$$m + s(x, y)q \leq (x \oplus y) + s(x, y)q = x + y < 2m + 2q - 1,$$

it follows that $s(x, y)q < m + 2q - 1$. Therefore, for any $0 \leq i < s(x, y)$, we have $iq = \wp(iq) \in C$ and the formula above for $f[x \mid y]$ is well defined.

Part 1. We prove in this step that the assignment in (44) extends to a morphism of complexes of $\mathbb{H}C$ -modules. This follows from Proposition 4.1, since one verifies recursively that

$$f[x_1 \mid \dots \mid x_n] \in \mathcal{R}_n(x_1 \oplus \dots \oplus x_n)$$

as follows: The case when $n = 0$ is obvious. When $n = 1$, it holds since $w_0 \in \mathcal{R}_1$ and $(x_1 - 1) \oplus \pi w_0 = (x_1 - 1) \oplus 1 = x_1$, and for $n = 2$ since $v_1 \in \mathcal{R}_2$ and

$$((x_1 \oplus x_2) - m) \oplus \pi v_1 = ((x_1 \oplus x_2) - m) \oplus m = x_1 \oplus x_2.$$

Then, for $n \geq 3$, induction gives

$$f[x_1 | \cdots | x_n] = f[x_1 | x_2] \bullet f[x_3 | \cdots | x_n] \in \mathcal{R}_2(x_1 \oplus x_2) \bullet \mathcal{R}_{n-2}(x_3 \oplus \cdots \oplus x_n) \subseteq \mathcal{R}_n(x_1 \oplus \cdots \oplus x_n).$$

Part 2. We prove now that $\partial f = f\partial$.

For a 1-cell $[x]$ of $\mathbf{B}(\mathcal{ZC})$, we have $\partial f[x] = x((x - 1)_* \partial w_0) \stackrel{(42)}{=} 0 = f\partial[x]$.

For a 2-cell $[x | y]$, we have

$$f\partial[x | y] = x_* f[y] - f[x \oplus y] + y_* f[x].$$

To compare with $\partial f[x | y]$, we shall distinguish three cases:

- *Case* $x + y < m + q$. In this case $\partial f[x | y] = 0$, and also

$$f\partial[x | y] = y((x + y - 1)_* w_0) - (x + y)((x + y - 1)_* w_0) + x((x + y - 1)_* w_0) = 0.$$

- *Case* $x + y \geq m + q$ and $x \oplus y = m$. Here, $(x - 1) \oplus y = m + q - 1 = x \oplus (y - 1)$. Then,

$$\begin{aligned} \partial f[x | y] &= \sum_{i=0}^{s(x,y)-1} (m + q)((iq \oplus (m + q - 1))_* w_0) - m((iq \oplus (m - 1))_* w_0) \\ &= (m + q)((m + q - 1)_* w_0) - m((m - 1)_* w_0) \\ &\quad + \sum_{i=1}^{s(x,y)-1} (m + q)((m + q - 1)_* w_0) - m((m + q - 1)_* w_0) \\ &= (m + q)((m + q - 1)_* w_0) - m((m - 1)_* w_0) + (s(x, y) - 1)q((m + q - 1)_* w_0) \\ &= (m + s(x, y)q)((m + q - 1)_* w_0) - m((m - 1)_* w_0) \\ &= (x + y)((m + q - 1)_* w_0) - m((m - 1)_* w_0) = f\partial[x | y]. \end{aligned}$$

- *Case* $x + y \geq m + q$ and $x \oplus y > m$. In this case, $(x - 1) \oplus y = (x \oplus y) - 1 = x \oplus (y - 1)$, whence

$$\begin{aligned} \partial f[x | y] &= \sum_{i=0}^{s(x,y)-1} ((x \oplus y) - m) \oplus iq)_* \partial v_1 = \sum_{i=0}^{s(x,y)-1} (m + q)((x \oplus y) - m) \oplus ((iq \oplus (m + q - 1))_* w_0) \\ &\quad - \sum_{i=0}^{s(x,y)-1} m(((x \oplus y) - m) \oplus (iq \oplus (m - 1))_* w_0) = \sum_{i=0}^{s(x,y)-1} (m + q)((x \oplus y) - 1)_* w_0 \\ &\quad - \sum_{i=0}^{s(x,y)-1} m((x \oplus y) - 1)_* w_0 = qs(x, y)((x \oplus y) - 1)_* w_0 \\ &= (y - (x \oplus y) + x)((x \oplus y) - 1)_* w_0 = f\partial[x | y]. \end{aligned}$$

For a 3-cell $[x | y | z]$, we have to prove that $f\partial[x | y | z] = 0$ or, equivalently, that

$$(45) \quad x_* f[y | z] + f[x | y \oplus z] = z_* f[x | y] + f[x \oplus y | z].$$

Since $x + (y \oplus z) = x \oplus y \oplus z + s(x, y \oplus z)q$, it follows that

$$x \oplus ((y \oplus z) - m) = ((x \oplus y \oplus z) - m) \oplus \wp(s(x, y \oplus z)q),$$

whenever $y \oplus z \geq m$. Then, we can write

$$\begin{aligned} x_* f[y \mid z] &= \begin{cases} 0, & \text{if } s(y, z) = 0, \\ (x \oplus ((y \oplus z) - m))_* \left(\sum_{i=0}^{s(y, z)-1} \wp(iq)_* v_1 \right), & \text{if } s(y, z) \geq 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } s(y, z) = 0, \\ ((x \oplus y \oplus z) - m)_* \left(\sum_{i=0}^{s(y, z)-1} \wp(s(x, y \oplus z)q + iq)_* v_1 \right), & \text{if } s(y, z) \geq 1. \end{cases} \end{aligned}$$

As

$$f[x \mid y \oplus z] = \begin{cases} 0, & \text{if } s(x, y \oplus z) = 0, \\ ((x \oplus y \oplus z) - m)_* \left(\sum_{i=0}^{s(x, y \oplus z)-1} \wp(iq)_* v_1 \right), & \text{if } s(x, y \oplus z) \geq 1, \end{cases}$$

one concludes the formula

$$x_* f[y \mid z] + f[x \mid y \oplus z] = \begin{cases} 0, & \text{if } s(y, z) = 0 = s(x, y \oplus z), \\ ((x \oplus y \oplus z) - m)_* \left(\sum_{i=0}^{s(y, z) + s(x, y \oplus z) - 1} \wp(iq)_* v_1 \right), & \text{otherwise.} \end{cases}$$

Similarly, one sees that

$$z_* f[x \mid y] + f[x \oplus y \mid z] = \begin{cases} 0, & \text{if } s(x, y) = 0 = s(x \oplus y, z), \\ ((x \oplus y \oplus z) - m)_* \left(\sum_{i=0}^{s(x, y) + s(x \oplus y, z) - 1} \wp(iq)_* v_1 \right), & \text{otherwise,} \end{cases}$$

and the equality in (45) follows by comparison using (39).

Finally, for a cell $[x \mid y \mid z \mid t \mid \cdots] = [x \mid y \mid z \mid \tau]$ of dimension higher than 3 we use the formulas

$$(46) \quad \partial[a \mid x \mid b] = [\partial[a \mid x] \mid b] + [a \mid \partial[x \mid b]]$$

which holds for any even chain a and any other chain b of $\mathbf{B}(\mathcal{Z}C)$, and

$$(47) \quad \partial(c \bullet d) = c \bullet \partial d,$$

which holds for any chains $c, d \in \mathcal{R}$. Thus, as we know that $f\partial[x \mid y \mid z] = 0$, induction gives

$$\begin{aligned} f\partial[x \mid y \mid z \mid \tau] &\stackrel{(46)}{=} f[\partial[x \mid y \mid z] \mid \tau] + f[x \mid y \mid \partial[z \mid \tau]] \\ &= f\partial[x \mid y \mid z] \bullet f[\tau] + f[x \mid y] \bullet f\partial[z \mid \tau] = f[x \mid y] \bullet \partial f[z \mid \tau] \\ &\stackrel{(47)}{=} \partial(f[x \mid y] \bullet f[z \mid \tau]) = \partial f[x \mid y \mid z \mid \tau] \end{aligned}$$

Part 3. Here we show that f preserves products. It is enough to prove that $f(\sigma \circ \tau) = f(\sigma) \circ f(\tau)$ for cells $\sigma = [x_1 \mid \cdots \mid x_n]$ and $\tau = [y_1 \mid \cdots \mid y_{n'}]$ of $\mathbf{B}(\mathcal{Z}C)$.

As in [8, page 99], a term $T = \pm[t_1 \mid \cdots \mid t_{n+n'}]$ in the shuffle product (18) of σ and τ is called *mixed* whenever there exists an index i such that t_{2i-1} is an x of σ and t_{2i} an y of τ , or vice versa. Choose the first index i for each mixed T , and let T' be the term obtained from T by interchanging t_{2i-1} with t_{2i} . Since $f[x, y]$ is symmetric,

$$\begin{aligned} f(T) &= f[t_1 \mid t_2] \bullet \cdots \bullet f[t_{2i-1} \mid t_{2i}] \bullet f[t_{2i+1} \mid \cdots] \\ &= f[t_1 \mid t_2] \bullet \cdots \bullet f[t_{2i} \mid t_{2i-1}] \bullet f[t_{2i+1} \mid \cdots] = f(T'). \end{aligned}$$

Since T and T' have opposite signs, the results cancel and $f(\sigma \circ \tau) = \sum f(T)$, with summation taken only over the unmixed terms, and where the sign of each term due the shuffle is always plus. If $n = 2r + 1$ and $n' = 2r' + 1$ are both odd, there are no unmixed terms, so $f(\sigma \circ \tau) = 0$ in agreement with the fact that $f(\sigma) \circ f(\tau) = 0$ (since $w_k \circ w_l = 0$). If $n = 2r$ and $n' =$

$2r'$ are both even, the unmixed terms T are obtained by taking all shuffles of the r pairs $(x_1, x_2), \dots, (x_{2r-1}, x_{2r})$ through the pairs $(y_1, y_2), \dots, (y_{2r'-1}, y_{2r'})$. For any such a shuffle

$$f(T) = f[x_1 | x_2] \bullet \cdots \bullet f[x_{2r-1} | x_{2r}] \bullet f[y_1 | y_2] \bullet \cdots \bullet f[y_{2r'-1}, y_{2r'}] = f(\sigma) \bullet f(\tau)$$

and the number of such shuffles is $\binom{r+r'}{r}$, hence

$$f(\sigma \circ \tau) = \binom{r+r'}{r} f(\sigma) \bullet f(\tau) = f(\sigma) \circ f(\tau),$$

as desired. For $n = 2r$ and $n' = 2r' + 1$, the unmixed terms T are as above but with the last argument $y_{2r'+1}$ always at the end. Hence, for each of them

$$f(T) = f[x_1 | x_2] \bullet \cdots \bullet f[x_{2r-1} | x_{2r}] \bullet f[y_1 | y_2] \bullet \cdots \bullet f[y_{2r'-1}, y_{2r'}] \bullet f[y_{2r'+1}] = f(\sigma) \bullet f(\tau),$$

and therefore $f(\sigma \circ \tau) = \binom{r+r'}{r} f(\sigma) \bullet f(\tau) = f(\sigma) \circ f(\tau)$. The remaining case $n = 2r + 1$ and $n' = 2r'$ is treated similarly. \square

Proposition 7.3. *A morphism $g : \mathcal{R} \rightarrow \mathbf{B}(\mathcal{ZC})$, of DGA-algebras over $\mathbb{H}C$, may be defined by the recursive formulas*

$$(48) \quad \begin{cases} gv_0 &= [\], \\ gw_k &= [gv_k | 1], \\ gv_{k+1} &= \sum_{t < m+q} (m+q-t-1)_* [gw_k | t] - \sum_{s < m} (m-s-1)_* [gw_k | s]. \end{cases}$$

Proof. Part 1. We show here that the assignment in (48) extends to a morphism of complexes of $\mathbb{H}C$ -modules. By Proposition 4.1, we have to verify that $gv_k \in \mathbf{B}(\mathcal{ZC})_{2k}(\wp(km))$ and $gw_k \in \mathbf{B}(\mathcal{ZC})_{2k+1}(\wp(km+1))$. Clearly $gv_0 = [\] \in \mathbf{B}(\mathcal{ZC})_0(0)$. Assume that $gv_k \in \mathbf{B}(\mathcal{ZC})_{2k}(\wp(km))$. Then, we have

$$gw_k = [gv_k | 1] \in \mathbf{B}(\mathcal{ZC})_{2k+1}(\wp(km) \oplus 1) = \mathbf{B}(\mathcal{ZC})_{2k+1}(\wp(km+1)),$$

as required. Moreover, for any $t < m+q$ and $s < m$,

$$(m+q-t-1)_* [gw_k | t], (m-s-1)_* [gw_k | s] \in \mathbf{B}(\mathcal{ZC})_{2k+2}(\wp((k+1)m)),$$

since

$$(m+q-t-1) \oplus \wp(km+1) \oplus t = \wp((k+1)m) = (m-s-1) \oplus \wp(km+1) \oplus s.$$

Whence $gv_{k+1} \in \mathbf{B}(\mathcal{ZC})_{2k+2}(\wp((k+1)m))$.

Part 2. Here we shall prove, as an auxiliary result, that

$$(49) \quad gv_k \circ [1] = gw_k, \quad gw_k \circ [1] = 0,$$

where $\circ = \circ_1$ is the shuffle product (18) of $\mathbf{B}(\mathcal{ZC})$. Clearly $gv_0 \circ [1] = [\] \circ [1] = [1] = [gv_0 | 1] = gw_0$. Assuming the result for gv_k , we have

$$gw_k \circ [1] = gv_k \circ [1] \circ [1] = gv_k \circ ([1 | 1] - [1 | 1]) = 0,$$

from where, in addition, it follows that, for any $t \in C$,

$$[gw_k | t] \circ [1] = [gw_k | t | 1] - [gw_k \circ [1] | t] = [gw_k | t | 1],$$

whence

$$\begin{aligned} gv_{k+1} \circ [1] &= \sum_{t < m+q} (m+q-t-1)_* [gw_k | t | 1] - \sum_{s < m} (m-s-1)_* [gw_k | s | 1] \\ &= [gv_{k+1} | 1] = gw_{k+1}. \end{aligned}$$

Part 3. We now prove recursively that $\partial g = g\partial$.

For argument w_0 is immediate: $\partial gw_0 = \partial[1] = 0$. For argument v_{k+1} , first observe that $\partial gw_k = 0$ gives, for any $t \in C$,

$$\begin{aligned}\partial[gv_k \mid t] &= \partial[gv_k \mid 1 \mid t] \stackrel{(46)}{=} [\partial[gv_k \mid 1] \mid t] + [gv_k \mid \partial[1 \mid t]] \\ &= [\partial gw_k \mid t] + [gv_k \mid \partial[1 \mid t]] = [gv_k \mid \partial[1 \mid t]] \\ &= 1_*[gv_k \mid t] - [gv_k \mid 1 \oplus t] + t_*[gv_k \mid 1] \\ &= 1_*[gv_k \mid t] - [gv_k \mid 1 \oplus t] + t_*gw_k.\end{aligned}$$

Then,

$$\begin{aligned}\partial gv_{k+1} &= \sum_{t < m+q} (m+q-t-1)_* \partial[gv_k \mid t] - \sum_{t < m} (m-t-1)_* \partial[gv_k \mid t] \\ &= \sum_{t < m+q-1} (m+q-t)_*[gv_k \mid t] - (m+q-t-1)_*[gv_k \mid 1+t] + (m+q-1)_*gw_k \\ &\quad + 1_*[gv_k \mid m+q-1] - [gv_k \mid m] + (m+q-1)_*gw_k \\ &\quad - \sum_{t < m} (m-t)_*[gv_k \mid t] - (m-t-1)_*[gv_k \mid 1+t] + (m-1)_*gw_k \\ &= -1_*[gv_k \mid m+q-1] + (m+q-1)((m+q-1)_*gw_k) \\ &\quad + 1_*[gv_k \mid m+q-1] - [gv_k \mid m] + (m+q-1)_*gw_k + [gv_k \mid m] - m(m-1)_*gw_k \\ &= (m+q)((m+q-1)_*gw_k) - m((m-1)_*gw_k) = g\partial v_{k+1}.\end{aligned}$$

And for argument w_{k+1} ,

$$\partial gw_{k+1} \stackrel{(49),(7)}{=} \partial gv_{k+1} \circ [1] = \left((m+q)((m+q-1)_*gw_k) - m((m-1)_*(gw_k)) \right) \circ [1] \stackrel{(49)}{=} 0.$$

Part 4. Here we show that g preserves products by proving that $g(a \circ b) = ga \circ gb$ for $a, b \in \{v_k, w_l\}$. For the case when $a = w_k$ and $b = w_l$, we have

$$gw_k \circ gw_l \stackrel{(49)}{=} gv_k \circ [1] \circ gw_l \stackrel{(49)}{=} 0 = g(w_k \circ w_l).$$

To prove the remaining cases, first observe that if $gv_k \circ gv_l = g(v_k \circ v_l)$ for some k and l , then

$$\begin{aligned}gw_k \circ gv_l &= gv_k \circ [1] \circ gv_l = gv_k \circ gv_l \circ [1] = g(v_k \circ v_l) \circ [1] = \\ &= \binom{k+l}{k} gv_{k+l} \circ [1] = \binom{k+l}{k} gw_{k+l} = g(w_k \circ v_l).\end{aligned}$$

Next, we show that $gv_k \circ gv_l = g(v_k \circ v_l)$ by induction. The case when $k = 0$ or $l = 0$ is immediate, since $gv_0 = []$. Now, using that, for any $t, s \in C$,

$$[gw_k \mid t] \circ [gw_l \mid s] \stackrel{(12)}{=} [[gw_k \mid t] \circ gw_l \mid s] + [gw_k \circ [gw_l \mid s], t],$$

we have

$$\begin{aligned}
gv_{k+1} \circ gv_{l+1} &= \sum_{s < m+q} (m+q-s-1)_* \left[\sum_{t < m+q} (m+q-t-1)_* [gw_k \mid t] \circ gw_l \mid s \right] \\
&\quad - \sum_{s < m+q} (m+q-s-1)_* \left[\sum_{t < m} (m-t-1)_* [gw_k \mid t] \circ gw_l \mid s \right] \\
&\quad + \sum_{t < m+q} (m+q-t-1)_* \left[gw_k \circ \sum_{s < m+q} (m+q-s-1)_* [gw_l \mid s] \mid t \right] \\
&\quad - \sum_{t < m+q} (m+q-t-1)_* \left[gw_k \circ \sum_{s < m} (m-s-1)_* [gw_l \mid s] \mid t \right] \\
&\quad - \sum_{t < m} (m-t-1)_* \left[gw_k \circ \sum_{s < m+q} (m+q-s-1)_* [gw_l \mid s] \mid t \right] \\
&\quad + \sum_{t < m} (m-t-1)_* \left[gw_k \circ \sum_{s < m} (m-s-1)_* [gw_l \mid s] \mid t \right] \\
&\quad - \sum_{s < m} (m-s-1)_* \left[\sum_{t < m+q} (m+q-t-1)_* [gw_k \mid t] \circ gw_l \mid s \right] \\
&\quad + \sum_{s < m} (m-s-1)_* \left[\sum_{t < m} (m-t-1)_* [gw_k \mid t] \circ gw_l \mid s \right],
\end{aligned}$$

and then, by induction,

$$\begin{aligned}
gv_{k+1} \circ gv_{l+1} &= \\
&= \sum_{s < m+q} (m+q-s-1)_* [gv_{k+1} \circ gw_l \mid s] + \sum_{t < m+q} (m+q-t-1)_* [gw_k \circ gw_{l+1} \mid t] \\
&\quad - \sum_{s < m} (m-s-1)_* [gv_{k+1} \circ gw_l \mid s] - \sum_{t < m} (m-t-1)_* [gw_k \circ gw_{l+1} \mid t] \\
&= \binom{k+l+1}{k+1} \left(\sum_{s < m+q} (m+q-s-1)_* [gw_{k+l+1} \mid s] - \sum_{s < m} (m-s-1)_* [gw_{k+l+1} \mid s] \right) \\
&\quad + \binom{k+l+1}{k} \left(\sum_{t < m+q} (m+q-t-1)_* [gw_{k+l+1} \mid t] - \sum_{t < m} (m-t-1)_* [gw_{k+l+1} \mid t] \right) \\
&= \binom{k+l+1}{k+1} gv_{k+l+2} + \binom{k+l+1}{k} gv_{k+l+2} = \binom{k+l+2}{k+1} gv_{k+l+2} \\
&= g(v_{k+1} \circ v_{l+1}).
\end{aligned}$$

□

Now, we are ready to establish the following key result.

Theorem 7.4. *The morphisms $f : \mathbf{B}(\mathcal{Z}C) \rightarrow \mathcal{R}$ and $g : \mathcal{R} \rightarrow \mathbf{B}(\mathcal{Z}C)$, as defined above, form a contraction.*

Proof. Part 1. We start by showing that the composite fg is the identity. Clearly $fgv_0 = f[\] = v_0$. Then, induction gives

$$\begin{aligned}
fgw_k &\stackrel{(49)}{=} f(gv_k \circ [1]) = fgv_k \circ f[1] = v_k \circ w_0 = w_k, \\
fgv_{k+1} &= \sum_{t < m+q} (m+q-t-1)_* f[gv_k | 1 | t] - \sum_{s < m} (m-s-1)_* f[gv_k | 1 | s] \\
&= \sum_{t < m+q} (m+q-t-1)_* (f[gv_k] \bullet f[1 | t]) - \sum_{s < m} (m-s-1)_* (f[gv_k] \bullet f[1 | s]) \\
&= v_k \bullet f[1 | m+q-1] = v_k \bullet v_1 = v_{k+1}.
\end{aligned}$$

Part 2. Here, we describe the composite gf . Clearly $gf[\] = [\]$ and $gf[x] = x((x-1)_*[1])$. For those 2-cells $[x | y]$ such that $x+y < m+q$ we have $gf[x | y] = 0$, and, as we prove below, the effect of gf on the 2-cells $[x | y]$ with $x+y \geq m+q$ is described by the formula

$$\begin{aligned}
(50) \quad gf[x | y] &= \sum_{t=x+y-m-q}^{m+q-1} (x+y-t-1)_*[1 | t] + \sum_{t=0}^{r-1} (m+r-t-1)_*[1 | t] \\
&\quad - \sum_{t=0}^{m-1} (m+r-t-1)_*[1 | t] + \sum_{i=1}^{s(x,y)-1} \sum_{t=(i-1)q+r}^{iq+r-1} (m+iq+r-t-1)_*[1 | t] \\
&\quad + \sum_{i=1}^{s(x,y)-1} \sum_{t=m}^{m+q-1} (m+iq+r-t-1)_*[1 | t],
\end{aligned}$$

where we write $x+y = m + s(x,y)q + r$ with $0 \leq r < q$ (so that $x \oplus y = m+r$). Concerning the two last terms, note that $(s(x,y)-1)q+r < m+q$ whenever $s(x,y) \geq 2$, since $m+s(x,y)q+r = x+y < 2m+2q$.

In effect, by definition of f and g , we have

$$gf[x | y] = \sum_{i=0}^{s(x,y)-1} \left(\sum_{t=0}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 | t] - \sum_{t=0}^{m-1} \wp(m+iq+r-t-1)_*[1 | t] \right).$$

Then, since for any $i \geq 1$ and $t < r$ is $\wp(m+(i+1)q+r-t-1) = \wp(m+iq+r-t-1)$, we see that

$$\begin{aligned}
gf[x | y] &= \sum_{t=r}^{m+q-1} (m+q+r-t-1)_*[1 | t] + \sum_{t=0}^{r-1} (m+r-t-1)_*[1 | t] - \sum_{t=0}^{m-1} (m+r-t-1)_*[1 | t] \\
&\quad + \sum_{i=1}^{s(x,y)-1} \left(\sum_{t=r}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 | t] - \sum_{t=r}^{m-1} \wp(m+iq+r-t-1)_*[1 | t] \right) \\
&= \sum_{t=0}^{r-1} (m+r-t-1)_*[1 | t] - \sum_{t=0}^{m-1} (m+r-t-1)_*[1 | t] \\
&\quad + \sum_{i=0}^{s(x,y)-1} \sum_{t=r}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 | t] - \sum_{i=1}^{s(x,y)-1} \sum_{t=r}^{m-1} \wp(m+iq+r-t-1)_*[1 | t],
\end{aligned}$$

from where (50) follows thanks to the equalities

$$\begin{aligned}
\sum_{t=r}^{m+q-1} \wp(m + (i+1)q + r - t - 1)_*[1 | t] &= \sum_{l=0}^{i-1} \sum_{t=lp+r}^{(l+1)q+r-1} (m + (l+1)q + r - t - 1)_*[1 | t] \\
&\quad + \sum_{t=iq+r}^{m+q-1} (m + (i+1)q + r - t - 1)_*[1 | t], \\
\sum_{t=r}^{m-1} \wp(m + iq + r - t - 1)_*[1 | t] &= \sum_{l=1}^{i-1} \sum_{t=(l-1)q+r}^{lq+r-1} (m + lq + r - t - 1)_*[1 | t] \\
&\quad + \sum_{t=(i-1)q+r}^{m-1} (m + iq + r - t - 1)_*[1 | t].
\end{aligned}$$

Finally, to complete the description of the composite gf , for generic cells $[x | y | \sigma]$ of dimensions greater than 2 we have the formula

$$(51) \quad gf[x | y | \sigma] = [gf[x, y] | gf[\sigma]].$$

In effect, as $gf[x | y | \sigma] = g(f[x, y] \bullet f[\sigma])$, by linearity, it suffices to observe that, for any $k \geq 1$,

$$g(v_1 \bullet w_k) = [gv_1 | gw_k], \quad g(v_1 \bullet v_k) = [gv_1 | gv_k],$$

or, equivalently, that $gw_{k+1} = [gv_1 | gw_k]$ and $gv_{k+1} = [gv_1 | gv_k]$. But these last equations are immediate for $k = 1$, and for higher k by a straightforward induction.

Part 3. We establish here a homotopy Φ from gf to the identity, which is determined by the recursive formulas

$$(52) \quad \begin{cases} \Phi[] = 0, \\ \Phi[x] = \sum_{t < x} (x - t - 1)_*[1 | t], \\ \Phi[x | y | \sigma] = [\Phi[x] | y | \sigma] + [gf[x | y] | \Phi[\sigma]]. \end{cases}$$

Since, for any $t < x$ in C , $(x - t - 1) \oplus 1 \oplus t = x$, we see that $\pi\Phi[x] = x$ and then, by recursion, that $\pi\Phi[x | y | \sigma] = x \oplus y \oplus \pi[\sigma]$. Hence, by Proposition 4.1, the formulas above determine an endomorphism of the complex of $\mathbb{H}C$ -modules $\mathbf{B}(\mathcal{Z}C)$, which is of differential degree $+1$.

Next, we prove that $\Phi : gf \Rightarrow id$ is actually a homotopy:

For a 1-cell $[x]$ is $\Phi\partial[x] = 0$, and

$$\partial\Phi[x] = \sum_{t < x} (x - t)_*[t] - (x - t - 1)_*[1 + t] + (x - 1)_*[1] = -[x] + x((x - 1)_*[1]) = -[x] + gf[x],$$

as required.

For a 2-cell $[x | y]$ we have

$$\begin{aligned}
(\partial\Phi + \Phi\partial)[x | y] &= \sum_{t < x} (x - t - 1)_*(1_*[t | y] - [1 + t, y] + [1 | t \oplus y] - y_*[1 | t]) \\
&\quad + \sum_{t < y} (x \oplus (y - t - 1))_*[1 | t] - \sum_{t < x \oplus y} ((x \oplus y) - t - 1)_*[1 | t] + \sum_{t < x} ((x - t - 1) \oplus y)_*[1 | t] \\
&= \sum_{t < x} (x - t)_*[t | y] - (x - t - 1)_*[1 + t | y] + \sum_{t < x} (x - t - 1)_*[1 | t \oplus y] - \sum_{t < x} ((x - t - 1) \oplus y)_*[1 | t] \\
&\quad + \sum_{t < y} (x \oplus (y - t - 1))_*[1 | t] - \sum_{t < x \oplus y} ((x \oplus y) - t - 1)_*[1 | t] + \sum_{t < x} ((x - t - 1) \oplus y)_*[1 | t] \\
&= -[x, y] + \sum_{t < x} (x - t - 1)_*[1 | t \oplus y] + \sum_{t < y} (x \oplus (y - t - 1))_*[1 | t] - \sum_{t < x \oplus y} ((x \oplus y) - t - 1)_*[1 | t].
\end{aligned}$$

If $s(x, y) = 0$ then, for any $t < x$, $t \oplus y = t + y$ and $x \oplus (y - t - 1) = x + y - t - 1 = (x \oplus y) - t - 1$. Therefore

$$\sum_{t < x} (x - t - 1)_* [1 \mid t + y] + \sum_{t < y} (x + y - t - 1)_* [1 \mid t] - \sum_{t < x+y} (x + y - t - 1)_* [1 \mid t] = 0,$$

and, since $gf[x \mid y] = 0$, it follows that $(\partial\Phi + \Phi\partial)[x \mid y] = -[x \mid y] + gf[x \mid y]$, as required.

If $s(x, y) > 0$, the composite $gf[x \mid y]$ has been computed in (50) and, writing as there $x + y = m + s(x, y)q + r$ with $0 \leq r < q$, we have

$$\begin{aligned} \sum_{t < x} (x - t - 1)_* [1 \mid t \oplus y] &= \sum_{l=1}^{s(x,y)-1} \sum_{\substack{t < x \\ m+lq \leq t+y < m+(l+1)q}} (x - t - 1)_* [1 \mid t + y - lq] \\ &+ \sum_{\substack{t < x \\ t+y < m+q}} (x - t - 1)_* [1 \mid t + y] + \sum_{\substack{t < x \\ m+s(x,y)q \leq t+y}} (x - t - 1)_* [1 \mid t + y - s(x, y)q]. \end{aligned}$$

Now, making the changes $u = t + y - lq$, $u = t + y$, and $u = t + y - s(x, y)q$ in the respective terms, and then renaming the u again by t , we obtain

$$\begin{aligned} \sum_{t < x} (x - t - 1)_* [1 \mid t \oplus y] &= \sum_{i=1}^{s(x,y)-1} \sum_{t=m}^{m+q-1} (m + iq + r - t - 1)_* [1 \mid t] \\ &+ \sum_{t=y}^{m+q-1} (x + y - t - 1)_* [1 \mid t] + \sum_{t=m}^{m+r-1} (m + r - t - 1)_* [1 \mid t]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sum_{t < y} (x \oplus (y - t - 1))_* [1 \mid t] &= \sum_{i=1}^{s(x,y)-1} \sum_{t=(i-1)q+r}^{iq+r-1} (m + iq + r - t - 1)_* [1 \mid t] \\ &+ \sum_{t=x+y-m-q}^{y-1} (x + y - t - 1)_* [1 \mid t] + \sum_{t=0}^{r-1} (m + r - t - 1)_* [1 \mid t], \end{aligned}$$

and

$$\sum_{t < x \oplus y} ((x \oplus y) - t - 1)_* [1 \mid t] = \sum_{t=0}^{m+r-1} (m + r - t - 1)_* [1 \mid t].$$

Hence, a direct comparison with (50) gives that $(\partial\Phi + \Phi\partial)[x \mid y] = -[x \mid y] + gf[x \mid y]$, as required.

Finally, we prove that $(\partial\Phi + \Phi\partial)(\tau) = -\tau + gf(\tau)$ if τ is a cell of dimension 3 or greater. To do so, previously observe that, for any generic cell γ of $\mathbf{B}(\mathcal{ZC})$, we have

$$(53) \quad \partial[gf[x \mid y] \mid \Phi(\gamma)] = [gf[x \mid y] \mid \partial\Phi(\gamma)].$$

To prove it, by linearity, it suffices to check that $\partial[gv_1 \mid 1 \mid \beta] = [gv_1 \mid \partial[1 \mid \beta]]$, for any generic cell β :

$$\begin{aligned} \partial[gv_1 \mid 1 \mid \beta] &\stackrel{(46)}{=} [\partial[gv_1 \mid 1] \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]] \\ &\stackrel{(48)}{=} [\partial gw_1 \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]] = [g\partial w_1 \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]] = [gv_1 \mid \partial[1 \mid \beta]]. \end{aligned}$$

Now, according to the definition in (52), on chains c of $\mathbf{B}(\mathcal{ZC})$ of dimensions 2 or greater, we can write $\Phi(c) = \Phi_1(c) + \Phi_2(c)$, where Φ_1 and Φ_2 are the morphisms of $\mathbb{H}C$ -modules given

on generic cells by $\Phi_1[x \mid y \mid \sigma] = [\Phi[x] \mid y \mid \sigma]$ and $\Phi_2[x \mid y \mid \sigma] = [gf[x \mid y] \mid \Phi(\sigma)]$. Then, for the generic cell $\tau = [x \mid y \mid z \mid \rho]$, as

$$\partial\tau = [\partial[x \mid y] \mid z \mid \rho] - [x \mid \partial[y \mid z \mid \rho]] = [\partial[x \mid y \mid z] \mid \rho] + [x \mid y \mid \partial[z \mid \rho]],$$

we have

$$\begin{aligned} \Phi\partial(\tau) &= \Phi_1[\partial[x \mid y] \mid z \mid \rho] - \Phi_1[x \mid \partial[y \mid z \mid \rho]] + \Phi_2[\partial[x \mid y \mid z] \mid \rho] + \Phi_2[x \mid y \mid \partial[z \mid \rho]] \\ &= [\Phi\partial[x \mid y] \mid z \mid \rho] - [\Phi[x] \mid \partial[y \mid z \mid \rho]] + [gf\partial[x \mid y \mid z] \mid \Phi[\rho]] + [gf[x \mid y] \mid \Phi\partial[z \mid \rho]] \\ &= [\Phi\partial[x \mid y] \mid z \mid \rho] - [\Phi[x] \mid \partial[y \mid z \mid \rho]] + [gf[x \mid y] \mid \Phi\partial[z \mid \rho]], \end{aligned}$$

since $f\partial[x \mid y \mid z] = 0$ by (45). Furthermore, by using (46) and (53), we have

$$\begin{aligned} \partial\Phi(\tau) &= \partial[\Phi[x] \mid y \mid z \mid \rho] + \partial[gf[x \mid y] \mid \Phi[z \mid \rho]] \\ &= [\partial\Phi[x \mid y] \mid z \mid \rho] + [\Phi[x] \mid \partial[y \mid z \mid \rho]] + [gf[x \mid y] \mid \partial\Phi[z \mid \rho]], \end{aligned}$$

whence, by the already proven above and induction on the dimension of ρ , we get

$$\begin{aligned} (\partial\Phi + \Phi\partial)(\tau) &= [\partial\Phi[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid \partial\Phi[z \mid \rho]] + [\Phi\partial[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid \Phi\partial[z \mid \rho]] \\ &= [(\partial\Phi + \Phi\partial)[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid (\partial\Phi + \Phi\partial)[z \mid \rho]] \\ &= [-[x \mid y] + gf[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid -[z \mid \rho] + gf[z \mid \rho]] \\ &= -[x \mid y \mid z \mid \rho] + [gf[x \mid y] \mid gf[z \mid \rho]] \stackrel{(51)}{=} -\tau + gf(\tau), \end{aligned}$$

as required.

This completes the proof of Theorem 7.4, since the conditions in (14) are easily verified. \square

If \mathcal{A} is any $\mathbb{H}C$ -module, by Proposition 5.3, the first level cohomology groups $H^n(C, 1; \mathcal{A})$ are precisely Leech cohomology groups $H_L^n(C, \mathcal{A})$. Hence, by Theorem 7.4, these can be computed as $H_L^n(C, \mathcal{A}) = H^n \text{Hom}_{\mathbb{H}C}(\mathcal{R}, \mathcal{A})$. Since, by Proposition 4.1, there are natural isomorphisms

$$\text{Hom}_{\mathbb{H}C}(\mathcal{R}_{2k}, \mathcal{A}) \cong \mathcal{A}(\wp(km)), \quad \text{Hom}_{\mathbb{H}C}(\mathcal{R}_{2k+1}, \mathcal{A}) \cong \mathcal{A}(\wp(km+1)).$$

we obtain the following already known result (see [4, Theorem 5.1] for a general result computing Leech cohomology groups for finite cyclic monoids).

Proposition 7.5 ([4, Corollary 5.6]). *Let $C = C_{m,q}$ be the cyclic monoid of index m and period q . Then, for any $\mathbb{H}C$ -module \mathcal{A} and any integer $k \geq 0$, there is a natural exact sequence of abelian groups*

$$0 \rightarrow H_L^{2k+1}(C, \mathcal{A}) \longrightarrow \mathcal{A}(\wp(km+1)) \xrightarrow{\partial} \mathcal{A}(\wp(km+m)) \longrightarrow H_L^{2k+2}(C, \mathcal{A}) \rightarrow 0,$$

where ∂ is given by $\partial(a) = (m+q)((m+q-1)_*a) - m((m-1)_*a)$.

Thus, for instance, if A is any abelian group, regarded as a constant $\mathbb{H}C$ -module, then the homomorphism $\partial : A \rightarrow A$ is multiplication by q , that is, $\partial(a) = qa$. Therefore, for all $k \geq 0$,

$$\begin{aligned} H_L^{2k+1}(C, A) &\cong \text{Ker}(q : A \rightarrow A), \\ H_L^{2k+2}(C, A) &\cong \text{Coker}(q : A \rightarrow A). \end{aligned}$$

We consider now the r th level cohomology groups of $C = C_{m,q}$ with $r \geq 2$. By Theorem 7.4 and an iterated use of Lemma 3.4 we conclude that the complexes of $\mathbb{H}C$ -modules $\mathbf{B}^r(\mathcal{Z}C)$ and $\mathbf{B}^{r-1}(\mathcal{R})$ are homotopy equivalent. Therefore, for any $\mathbb{H}C$ -module \mathcal{A} , there are natural isomorphisms

$$H^n(C, r, \mathcal{A}) \cong H^n(\text{Hom}_{\mathbb{H}C}(\mathbf{B}^{r-1}(\mathcal{R}), \mathcal{A})).$$

An analysis of the complexes $\mathbf{B}^{r-1}(\mathcal{R})$ tell us that $\mathbf{B}^{r-1}(\mathcal{R})_n = 0$ for $0 < n < r$, and that we have the diagram of suspensions

$$\begin{array}{ccccccccc}
 \mathcal{R}_4 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \mathcal{R}_2 & \longrightarrow & \mathcal{R}_1 & \longrightarrow & 0 \\
 \downarrow \text{S} & & \downarrow \text{S} & & \downarrow \text{S} & & \downarrow \text{S} & & \\
 \mathbf{B}(\mathcal{R})_5 & \longrightarrow & \mathbf{B}(\mathcal{R})_4 & \longrightarrow & \mathbf{B}(\mathcal{R})_3 & \longrightarrow & \mathbf{B}(\mathcal{R})_2 & \longrightarrow & 0 \\
 \downarrow \text{S} & & \downarrow \text{S} & & \downarrow \text{S} & & \downarrow \text{S} & & \\
 \mathbf{B}^2(\mathcal{R})_6 & \longrightarrow & \mathbf{B}^2(\mathcal{R})_5 & \longrightarrow & \mathbf{B}^2(\mathcal{R})_4 & \longrightarrow & \mathbf{B}^2(\mathcal{R})_3 & \longrightarrow & 0
 \end{array}$$

where

- $\mathbf{B}(\mathcal{R})_4$ is the free $\mathbb{H}C$ -module on the binary set consisting of the suspension of the 3-cell w_1 of \mathcal{R} and the 4-cell

$$[w_0 | w_0]$$

with $\pi[w_0 | w_0] = \wp(2)$, whose differential is $\partial([w_0 | w_0]) = w_0 \circ w_0 = 0$,

- $\mathbf{B}(\mathcal{R})_5$ is the free $\mathbb{H}C$ -module on the set consisting of the suspension of the 4-cell v_2 of \mathcal{R} together the 5-cells

$$[w_0 | v_1], [v_1 | w_0]$$

with $\pi[w_0 | v_1] = m \oplus 1 = \pi[v_1 | w_0]$, and whose differential is

$$\begin{aligned}
 \partial[w_0 | v_1] &= w_1 - (m+q)((m+q-1)_*[w_0 | w_0]) + m((m-1)_*[w_0 | w_0]), \\
 \partial[v_1 | w_0] &= -w_1 - (m+q)((m+q-1)_*[w_0 | w_0]) + m((m-1)_*[w_0 | w_0]).
 \end{aligned}$$

- $\mathbf{B}^2(\mathcal{R})_6$ is the free $\mathbb{H}C$ -module on the set consisting of the double suspension of the 4-cell v_2 of \mathcal{R} , the suspension of the 5-cells $[w_0 | v_1]$ and the $[v_1 | w_0]$ of $\mathbf{B}(\mathcal{R})_5$, and the 6-cell

$$[w_0 \parallel w_0]$$

with $\pi[w_0 \parallel w_0] = \wp(2)$, whose differential is

$$\partial[w_0 \parallel w_0] = 0.$$

Then, by Proposition 4.1, there are natural isomorphisms

$$\begin{aligned}
 \text{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_2, \mathcal{A}) &\cong \mathcal{A}(1), \quad \text{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_4, \mathcal{A}) \cong \mathcal{A}(m \oplus 1) \times \mathcal{A}(\wp(2)), \\
 \text{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_3, \mathcal{A}) &\cong \mathcal{A}(m), \quad \text{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_5, \mathcal{A}) \cong \mathcal{A}(\wp(2m)) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(m \oplus 1), \\
 \text{Hom}_{\mathbb{H}C}(\mathbf{B}^2(\mathcal{R})_6, \mathcal{A}) &\cong \mathcal{A}(\wp(2m)) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(\wp(2)).
 \end{aligned}$$

In these terms the truncated complex $\text{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R}), \mathcal{A})$ is written as

$$(54) \quad 0 \rightarrow \mathcal{A}(1) \xrightarrow{\partial^1} \mathcal{A}(m) \xrightarrow{\partial^2} \mathcal{A}(m \oplus 1) \times \mathcal{A}(\wp(2)) \xrightarrow{\partial^3} \mathcal{A}(\wp(2m)) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(m \oplus 1),$$

where the coboundaries are given by

$$\partial^1(a) = -(m+q)((m+q-1)_*a) + m((m-1)_*a),$$

$\partial^2 = 0$ is the morphism zero, and

$$\begin{aligned}
 \partial^3(a, b) &= \left(-(m+q)((m+q-1)_*a) + m((m-1)_*a), \right. \\
 &\quad \left. a - (m+q)((m+q-1)_*b) + m((m-1)_*b), \right. \\
 &\quad \left. -a - (m+q)((m+q-1)_*b) + m((m-1)_*b) \right),
 \end{aligned}$$

while the truncated complex $\text{Hom}_{\mathbb{H}C}(\mathbf{B}^2(\mathcal{R}), \mathcal{A})$ is written as

$$(55) \quad 0 \rightarrow \mathcal{A}(1) \xrightarrow{\partial^1} \mathcal{A}(m) \xrightarrow{\partial^2} \mathcal{A}(m \oplus 1) \times \mathcal{A}(\wp(2)) \xrightarrow{\partial^3} \mathcal{A}(\wp(2m)) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(m \oplus 1) \times \mathcal{A}(\wp(2)),$$

where ∂^1 and ∂^2 are the same as above whereas ∂^3 acts by

$$\begin{aligned} \partial^3(a, b) = & \left((m+q)((m+q-1)_*a) - m((m-1)_*a), \right. \\ & -a + (m+q)((m+q-1)_*b) - m((m-1)_*b), \\ & \left. a + (m+q)((m+q-1)_*b) - m((m-1)_*b), 0 \right). \end{aligned}$$

Then, as an immediate consequence of (54) and (55), we have

Theorem 7.6. *Let $C = C_{m,q}$ be the cyclic monoid of index m and period q . Then, for any $\mathbb{H}C$ -module \mathcal{A} , there is a natural exact sequence of abelian groups*

$$0 \rightarrow H^2(C, 2; \mathcal{A}) \rightarrow \mathcal{A}(1) \xrightarrow{\partial} \mathcal{A}(m) \rightarrow H^3(C, 2; \mathcal{A}) \rightarrow 0$$

where $\partial(a) = (m+q)((m+q-1)_*a) - m((m-1)_*a)$, and natural isomorphisms

$$H^4(C, 2; \mathcal{A}) \cong H^5(C, 3; \mathcal{A}) \cong \left\{ b \in \mathcal{A}(\wp(2)) \left| \begin{array}{l} (m+q)^2 \wp(2m+q-2)_*b = m^2 \wp(2m-2)_*b, \\ 2(m+q)(m+q-1)_*b = 2m(m-1)_*b, \end{array} \right. \right\}.$$

Note that in the case when the cyclic monoid is of index $m = 1$, the above description of $H^4(C, 2; \mathcal{A})$ adopts the simpler form

$$H^4(C, 2; \mathcal{A}) \cong \left\{ b \in \mathcal{A}(\wp(2)) \left| \begin{array}{l} (q+1)^2 q_*b = b, \\ 2(q+1)q_*b = 2b, \end{array} \right. \right\},$$

while when $m \geq 2$,

$$H^4(C, 2; \mathcal{A}) \cong \left\{ b \in \mathcal{A}(\wp(2)) \left| \begin{array}{l} (2mq+q^2)\wp(2m-2)_*b = b, \\ 2(m+q)(m+q-1)_*b = 2m(m-1)_*b, \end{array} \right. \right\}.$$

Corollary 7.7. *For any finite cyclic monoid C , any integer $r \geq 1$, and any $\mathbb{H}C$ -module \mathcal{A} , there are natural isomorphisms*

$$H^{r+1}(C, r; \mathcal{A}) \cong H_L^2(C, \mathcal{A}) \cong H_G^2(C, \mathcal{A}).$$

Proof. A direct comparison of the exact sequence in Theorem 7.6 with the sequence in Proposition 7.5, for the case when $k = 0$, gives $H^3(C, 2; \mathcal{A}) \cong H_L^2(C, \mathcal{A})$. Then, the result follows since $H^3(C, 2; \mathcal{A}) \cong H_G^2(C, \mathcal{A})$ by Proposition 5.11, and $H^{r+1}(C, r; \mathcal{A}) \cong H^3(C, 2; \mathcal{A})$ by Corollary 5.8. \square

Corollary 7.8. *For any finite cyclic monoid C , any integer $r \geq 2$, and any $\mathbb{H}C$ -module \mathcal{A} , there are natural isomorphisms*

$$H^{r+2}(C, r; \mathcal{A}) \cong H_C^3(C, \mathcal{A}).$$

Proof. By Corollary 5.9, $H^{r+2}(C, r; \mathcal{A}) \cong H^5(C, 3; \mathcal{A})$, for any $r \geq 3$. Since, by Theorem 7.6, $H^5(C, 3; \mathcal{A}) \cong H^4(C, 2; \mathcal{A})$, the result follows by Proposition 5.12. \square

For instance, if A is any abelian group viewed as a constant $\mathbb{H}C$ -module, then $H^4(C, 2; A)$ is isomorphic to the subgroup of A consisting of those elements b such that

$$\left| \begin{array}{l} (m+q)^2 b = m^2 b, \\ 2qb = 0, \end{array} \right| \Leftrightarrow \left| \begin{array}{l} (2mq+q^2)b = 0, \\ 2qb = 0, \end{array} \right| \Leftrightarrow \left| \begin{array}{l} q^2 b = 0, \\ 2qb = 0, \end{array} \right| \Leftrightarrow (2q, q^2)b = 0,$$

where $(2q, q^2) = q(2, q)$ is the greatest common divisor of 2 and q . This leads to the following isomorphism, which is analogous to the proven by Eilenberg- Mac Lane for the third abelian cohomology group of the cyclic group C_q with coefficients in A [8, §21].

Corollary 7.9. *For any finite cyclic monoid C , any integer $r \geq 2$, and any abelian group A , there is a natural isomorphism*

$$H^{r+2}(C, r; A) \cong \text{Hom}_{\mathbf{Ab}}(\mathbb{Z}/(2q, q^2)\mathbb{Z}, A).$$

7.2. Cohomology of the infinite cyclic monoid.

In this subsection we focus on the additive monoid of natural numbers $C_\infty = \mathbb{N}$. As before, we start by introducing a commutative DGA-algebra over $\mathbb{H}C_\infty$, \mathcal{R} , simpler than $\mathbf{B}(\mathcal{Z}C_\infty)$.

For each integer $k = 0, 1, \dots$, let us choose unitary sets over C_∞ , $\{w_0\}$ and $\{v_k\}$, with $\pi w_0 = 1$ and $\pi v_k = k$. Then,

$$\begin{cases} \mathcal{R}_0 &= \text{the free } \mathbb{H}C_\infty\text{-module on } \{v_0\}, \\ \mathcal{R}_1 &= \text{the free } \mathbb{H}C_\infty\text{-module on } \{w_0\}, \\ \mathcal{R}_n &= 0, \quad n \geq 2 \end{cases}$$

The differential $\partial = 0$ is zero. The augmentation is the canonical isomorphism $\mathcal{R}_0 \cong \mathbb{Z}$, and the multiplication on \mathcal{R} is by determined by the rules $v_0 \circ v_0 = v_0$, $v_0 \circ w_0 = w_0$ and $w_0 \circ w_0 = 0$.

Theorem 7.10. *There are DGA-algebra morphisms $f : \mathbf{B}(\mathcal{Z}C_\infty) \rightarrow \mathcal{R}$ and $g : \mathcal{R} \rightarrow \mathbf{B}(\mathcal{Z}C_\infty)$, determined by the formulas*

$$\begin{cases} f[\] &= v_0, \\ f[x] &= x((x-1)_*w_0) \end{cases} \quad \begin{cases} gv_0 &= [\], \\ gw_0 &= [1], \end{cases}$$

which form a contraction.

Proof. It is plain to see that above assignments determine well defined morphisms of DGA-algebras over $\mathbb{H}C_\infty$. To prove that they form a contraction, we limit ourselves to describe the homotopy $\Phi : gf \Rightarrow id$, by the formula below, because the details are parallel and much more simpler than those in the proof of Theorem 7.4.

$$\begin{cases} \Phi[\] &= 0, \\ \Phi[x] &= \sum_{0 \leq t < x} (x-t-1)_*[1 \mid t], \\ \Phi[x \mid \sigma] &= [\Phi[x] \mid \sigma], \end{cases}$$

with σ any cell of dimension greater than 1. \square

By Proposition 5.3, there are isomorphisms $H^n(C_\infty, 1; \mathcal{A}) \cong H_L^n(C_\infty, \mathcal{A})$, for any $\mathbb{H}C_\infty$ -module \mathcal{A} . Then, as consequence of Theorem 7.10, we recover the computation by Leech of the cohomology groups of the monoid C_∞ [16, Theorem 6.8].

Proposition 7.11. *For any $\mathbb{H}C_\infty$ -module \mathcal{A} , there are natural isomorphisms*

$$H_L^0(C_\infty, \mathcal{A}) \cong \mathcal{A}(0), \quad H_L^1(C_\infty, \mathcal{A}) \cong \mathcal{A}(1),$$

and for every $n \geq 2$, $H_L^n(C_\infty, \mathcal{A}) = 0$.

We now pay attention to the second level cohomology groups of C_∞ . By Theorem 7.10 and Lemma 3.4, $H^n(C_\infty, 2; \mathcal{A}) \cong H^n(\text{Hom}_{\mathbb{H}C_\infty}(\mathbf{B}(\mathcal{R}), \mathcal{A}))$. An analysis of $\mathbf{B}(\mathcal{R})$ tell us that

$$\begin{cases} \mathbf{B}(\mathcal{R})_{2k} &= \text{the free } \mathbb{H}C_\infty\text{-module on } \{v_k\}, \\ \mathbf{B}(\mathcal{R})_{2k+1} &= 0, \end{cases}$$

where, recall, $\pi v_k = k$; the augmentation is the canonical isomorphism $\mathbf{B}(\mathcal{R})_0 \cong \mathbb{Z}$ and the product is given by

$$v_k \circ v_l = \binom{k+l}{k} v_{k+l}.$$

Hence,

Proposition 7.12. *For any $\mathbb{H}C_\infty$ -module \mathcal{A} , and any integer $k \geq 0$,*

$$H^{2k}(C_\infty, 2; \mathcal{A}) \cong \mathcal{A}(k), \quad H^{2k+1}(C_\infty, 2; \mathcal{A}) = 0.$$

From Corollary 5.8, it follows that

Corollary 7.13. *For any $\mathbb{H}C_\infty$ -module \mathcal{A} , and any integer $r \geq 2$,*

$$H^{r+1}(C_\infty, r; \mathcal{A}) = 0.$$

We finish by specifying the 3rd level 5-cohomology group of C_∞ .

Proposition 7.14. *For any $\mathbb{H}C_\infty$ -module \mathcal{A} , and any integer $r \geq 3$, there is a natural isomorphism*

$$H^{r+2}(C_\infty, r; \mathcal{A}) \cong \{a \in \mathcal{A}(2) \mid 2a = 0\}.$$

Proof. By Corollary 5.9, $H^{r+2}(C_\infty, r; \mathcal{A}) \cong H^5(C_\infty, 3; \mathcal{A})$. An analysis of $\mathbf{B}^2(\mathcal{R})$ tell us that $\mathbf{B}^2(\mathcal{R})_4 = \mathbf{B}(\mathcal{R})_3 = 0$, $\mathbf{B}^2(\mathcal{R})_5 = \mathbf{B}(\mathcal{R})_4$ is the free $\mathbb{H}C_\infty$ -module on $\{v_2\}$, where $\pi v_2 = 2$, $\mathbf{B}^2(\mathcal{R})_6$ is the free $\mathbb{H}C_\infty$ -module on $\{[v_1 \parallel v_1]\}$, with $\pi[v_1 \parallel v_1] = 2$, and the differential is

$$\partial[v_1 \parallel v_1] = -2v_2.$$

Whence, for any $\mathbb{H}C_\infty$ -module \mathcal{A} , $H^5(C_\infty, 3; \mathcal{A}) \cong \{a \in \mathcal{A}(2) \mid 2a = 0\}$. \square

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